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Comparing 2k and 3k Factorial Design Using Information Based Criteria

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ABSTRACT

This research aimed at identifying the better model in between 2^k and 3^k factorial designs for k=2 and 3 using information base criterion. The codes ± 1were used to obtain the design matrix of 2^k factorial designs while ± 1, 0 and 0, 1, 2 provided the elements of the design matrix for 3^k factorial designs. While, A-, D-, E- and G- optimality for each of the Design models were utilized in the analysis with the aid of R- package to determine the optimal values and the efficiency of each model. From our results, it observed that the 2^k designs produced efficient estimates of the model parameters when A-, D-, E-, and G- Optimality are considered. Therefore, 2^k factorial designs are more efficient than 3^k factorial designs for all values of k under consideration using information based criteria. It recommended that, for study of factorial designs, 2^k design is suitable in A-, D-, E-, and G-optimality for k = 2 and 3.

Keywords: 2^k and 3^k factorial designs, optimal designs

INTRODUCTION

The design of experiments is an efficient method for planning experiments, so that, the data obtained can be analysed to yield valid and effective conclusions. The method for conducting designed of experiments begins with determining the objectives of an experiment and selecting the process factors for the study. A designed of experiment requires establishing a detailed experimental plan in advance of conducting the experiment, which results in a streamlined approach in the data collection stage. Appropriately choosing experimental designs maximizes the amount of information that can be obtained for a given amount of experimental effort (Yong Guo, 2006).

The statistical design of the experiment deals with assigning the treatment combinations of interest to the available experimental units. For a given research question, a number of several experimental designs can be considered and the optimal design is the one that ensures efficient estimator of the model parameters. The optimal design helps to make a valid conclusion of the experiment.

By an optimality of a design, we mean a design that is ''best'' with respect to the same criteria. Optimal design is the design that achieves some target of our interest. The word ''optimal'' depending on the situation when used can mean effective, minimum variance minimum bias etc. optimal has also several interpretations in the context of experimental design

Finding an optimal design is considered an essential topic in the context of experimental studies that deserves special attention from many researchers who are interested in data analysis in several fields such as agriculture, engineering, marketing, and pharmacy among others.

According to Emery, and Nenarokomov. (1998). Optimal experimental design is the definition of the conditions under which an experiment is to be conducted in order to maximize the accuracy with which the results are obtained.

Optimal Design of Experiments offers a rare blend of linear algebra, convex analysis, and statistics. The optimal design for statistical experiments is first formulated as a concave matrix optimization problem. Using tools from convex analysis, the problem is solved generally for a wide class of optimality criteria such as D-, A-, or E-optimality. Friedrich pukelsheim, (2006).

According to Byran Smucker, (2018), optimal design optimizes a numerical criterion, which typically relates to the variance or other statistically relevant properties of the design, and uses as input the number of runs, the factors and their possible levels, block structure (if any), and a hypothesized form of the relationship between the response and the factors. In the design of experiments, optimal designs (or optimum designs) are a class of experimental designs that are optimal with respect to some statistical criterion.

An optimality criterion is a single-valued measure that determines how good a design is, and it is maximized or minimized by an optimal design.

There are many optimal criteria. These criteria are sometime called alphabetical criteria. It can be classified into four types namely: Information based criteria, Distance based criteria, Compound based criteria and other type criteria (Rady, Abd El-Monsef and Seyam, 2019).

This study tend to compare the two special cases of factorial design, that is 2^K and 3^K factorial designs using information based criteria to find out which design is efficient and at which condition(s).

An information criterion is a measure of the quality of a statistical model. It takes into account the complexity of the model. Information criteria are used to compare alternative models fitted to the same data set.

METHODOLOGY

The method adopted the existing coded level of ±1 and (±1,0 and 0,1,2) as the elements for generating the design matrix for 2^k and 3^k factorial respectively.

The 2^k Factorial Design:

The 2^k factorial design is a special case of the general factorial design; k factors are being studied, all at 2 levels (i.e. high, referred as "+" or "+1", and low, referred as "-"or "-1").

The 2^k Factorial Design consists of *k* factors each at only two levels and is a special case of the full factorial design with 2^k observations per replication.

In factorial design, we use capital letter to denote the effect i.e A and B = A and B effect and AB = interaction of A and B.

For (k = 2) that is 2^2 factorial design:

The linear statistical model is as follows:

$$Yijk = \mu + \alpha i + \beta j + (\alpha\beta)ij + \varepsilon_{ijk} \begin{cases} i = 1, 2, \dots, a \\ j = 1, 2, \dots, b \\ k = 1, 2, \dots, n \end{cases}$$

Where:

$Yijk$ Represent the observation taken under the i^{th} level of factor A and the j^{th} level of factor B in the k^{th} replicate.

μ Represent the overall mean effect.

α_i Represent the i^{th} effect of factor A.

β_j Represent the j^{th} level of effect of factor B.

$(\alpha\beta)_{ij}$ Represent the effect of the interaction between α_i and β_j.

ε_{ijk} Represent the random error component.

The linear regression of 2^2 is $y(x_1x_2) = \beta_0 + \beta_1x_1 + \beta_2x_2 + \beta_{12}x_1x_2 +$ E

So that $E(y) = \beta_0 + \beta_1x_1 + \beta_2x_2 + \beta_{12}x_1x_2$.

To make this representation explicit it is useful to write the above formula in matrix notation as.

$Y = X\beta +$ E

Where Y is an (n × 1) vector of observation, X is an (n × p) design matrix, β is a (p × 1) vector of the error regression parameters and E is the error.

Table 3.1. Algebraic sign for finding the effect of 2^2 design.

Treatment Combination	Factorial effect 1	A	B	AB
	+	–	–	+
a	+	+	–	–
b	+	–	+	–
ab	+	+	+	+

In the design matrix of 2^2, + and – are represent by +1 and -1 respectively for high and low level of each factor.

The matrix form of 2^2 is

$$x = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

(k=3) that is 2^3 Factorial design:

The linear statistical model is as follows:

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijl} + \mathrm{E}_{ijkl} \begin{cases} i = 1,2,\ldots,a \\ j = ,1,2,\ldots,b \\ k = 1,2,\ldots,c \\ l = 1,2,\ldots,n \end{cases}$$

Where:

y_{ijkl} Represent the observation i^{th} level of factor A and j^{th} level of factor B and k^{th} level of factor C and the l^{th} replicate.

μ Represent the overall mean effect.

α_i Represent the true effect of the ith level of factor A,

β_j Represent the true effect of the jth level of factor B,

γ_k Represent the true effect of the kth level of factor C

$(\alpha\beta)_{ij}$ Represent the effect of the interaction between α_i and β_j

$(\alpha\gamma)_{ik}$ Represent the effect of interaction between α_i and γ_k

$(\beta\gamma)_{jk}$ Represent the effect of interaction between β_j and γ_k

$(\alpha\beta\gamma)_{ijl}$ Represent the effect of interaction between α_i, β_j and γ_k

E_{ijkl} Represent the random error component.

The linear regression of 2^3 is

$$y(x_1x_2x_3) = \beta_{00} + \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_{12}x_1x_2 + \beta_{13}x_1x_3 + \beta_{23}x_2x_3 + \beta_{123}x_1x_2x_3 + \mathrm{E}$$

$$E(y) = \beta_{00} + \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_{12}x_1x_2 + \beta_{13}x_1x_3 + \beta_{23}x_2x_3 + \beta_{123}x_1x_2x_3 .$$

To make this representation explicit it is useful to write the above formula in matrix notation as.

$$Y = X\beta + \text{E}$$

Where Y is an (n × 1) vector of observation, X is an (n × p) design matrix, β is a (p × 1) vector of the error regression parameters and E is the error.

Table 3.2. Algebraic sign for finding the effect in the 2^3 design. Factorial effect

Treatment Combination	1	A	B	AB	C	AC	BC	ABC
1	+	-	-	+	-	+	+	-
a	+	+	-	-	-	-	+	+
b	+	-	+	-	-	+	-	+
ab	+	+	+	+	-	-	-	-
c	+	-	-	+	+	-	-	+
ac	+	+	-	-	+	+	-	-
bc	+	-	+	-	+	-	+	-
abc	+	+	+	+	+	+	+	-

The design matrix for 2^3 designs is

$$X = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The 3^k factorial design.

We can consider the analysis of a *k*-factor factorial design where each factor has 3 levels: 0 (low), 1 (intermediate), and 2 (high). We now have 3^k treatments which we denote by *k* digit combinations of 0, 1, and 2 instead of the standard notation (1), *a*, *b, ab* ... For example, for a 3^2 design the treatments are 00, *10*, *20*, *01*, 11, *21*, *02*, 12, *22*. The treatment 01, for instance, is the combination of the low level of factor *A* and the intermediate level of factor *B*. Computations of effects and sums of squares are direct extensions of the 2^k case.

For k = 2 that is 3^2 Factorial design.

The statistical linear regression is:

$$y(x_1x_2) = \beta_{00} + \beta_{10}x_1 + \beta_{01}x_2 + \beta_{11}x_1x_2 + \beta_{12}x_1x_2^2 + \beta_{20}x_1^2 + \beta_{02}x_2^2 + \beta_{21}x_1^2x_2 + \beta_{22x}x_2^2x_2^2 + \mathrm{E}$$

Then

$$E(Y) = \beta_{00} + \beta_{10}x_1 + \beta_{01}x_2 + \beta_{11}x_1x_2 + \beta_{12}x_1x_2^2 + \beta_{20}x_1^2 + \beta_{02}x_2^2 + \beta_{21}x_1^2x_2 + \beta_{22x}x_2^2x_2^2$$

Where

β_{00} Represent both factors at low level denoted by (1),

$\beta_{10}x_1$ Represent factor A at intermediate level denoted by A_L

$\beta_{01}x_2$ Represent factor B at intermediate level denoted B_L

$\beta_{11}x_1x_2$ Represent the interaction between factor A and B at intermediate level denoted byA_LB_L,

$\beta_{20}x_1^2$ Represent factor A at high level denoted byA_Q,

$\beta_{02}x_2^2$ Represent factor B at high level denoted by B_Q

$\beta_{21}x_1^2x_2$ Represent the interaction between factor A at high level and factor B at intermediate level denoted byA_QB_L,

$\beta_{12}x_1x_2^2$ Represent the interaction between factor A at intermediate level and factor B at high level denoted byA_LB_Q,

$\beta_{22}x_1^2x_2^2$ Represent both factor at high level denoted byA_QB_Q.

ε Is the error component.

To make this representation explicit it is useful to write the above formula in matrix notation as.

$$Y = X\beta + \mathrm{E}$$

Where Y is an (n × 1) vector of observation, X is an (n × p) design matrix, β is a (p × 1) vector of the error regression parameters and E is the error.

Table 3.3. The design matrix of 3^2 Factorial experiment.

Run	(1)	β_{01}	β_{02}	β_{10}	β_{11}	β_{12}	β_{20}	β_{21}	β_{22}
1	1	-1	1	-1	1	-1	1	-1	1
2	1	0	0	-1	0	0	1	0	0
3	1	1	1	-1	-1	-1	1	1	1
4	1	-1	1	0	0	0	0	0	0
5	1	0	0	0	0	0	0	0	0
6	1	1	1	0	0	0	0	0	0
7	1	-1	1	1	-1	1	1	-1	1
8	1	0	0	1	0	0	1	0	0
9	1	1	1	1	1	1	1	1	1

The design matrix of 3^2 Factorial design

$$X = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

For (k=3) that is 3^3 Factorial design:

The statistical linear regression is:

$$Y(x_1x_2x_3) = \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_{12}x_1x_2 + \beta_{13}x_1x_3 + \beta_{23}x_2x_3 + \beta_{122}x_1x_2^2 + \beta_{133}x_1x_3^2 + \beta_{233}x_2x_3^2 + \beta_{123}x_1x_2x_3$$
$$+\beta_{1223}x_1x_2^2x_3 + \beta_{1233}x_1x_2x_3^2 + \beta_{12233}x_1x_2^2x_3^2$$

Sothat

$$E(Y) = \beta_1x_1 + \beta_2x_2 + \beta_3x_3 + \beta_{12}x_1x_2 + \beta_{13}x_1x_3 + \beta_{23}x_2x_3 + \beta_{122}x_1x_2^2 + \beta_{133}x_1x_3^2 + \beta_{233}x_2x_3^2 + \beta_{123}x_1x_2x_3$$
$$+\beta_{1223}x_1x_2^2x_3 + \beta_{1233}x_1x_2x_3^2 + \beta_{12233}x_1x_2^2x_3^2$$

to make this representation explicit it is useful to write the above formula in matrix notation as.

$$Y = X\beta + \text{E}$$

Where Y is an (n × 1) vector of observation, X is an (n × p) design matrix, β is a (p × 1) vector of the error regression parameters and E is the error.

Since a 3^3 design is a special case of a multi-way layout we can compute the sum of squares for main effects *A*, *B*, *C*, interactions *A*×*B*, *A*×*C*, *B*×*C* and *A*×*B*×*C*.

These components can be manipulated using a symbolic notation. Consider the interaction between AB and AB^2. Thus AB×AB^2, which gives us A^2B^3 which using modular (3) arithmetic gives us $A^2B^0 = A^2 = (A^2)^2 = A$. Therefore, the interaction between these two terms gives us the main effect. If we wanted to look at a term such as A^2B or A^2B^2, we would reduce it by squaring it which would give us: $(A^2B)^2 = AB^2$ and likewise $(A^2B^2)^2 = AB$. We never include a component that has an exponent higher than one on the first letter because by squaring it we obtain an equivalent component. This is just a way of partitioning the treatment combinations and these labels are just an arbitrary identification of them.

A×*B* has 4 degrees of freedom, it also *A*×*B* has two components denoted by *AB* and AB^2, A×C has two components denoted AC and AC^2 and B×C also has two components denoted by BC and BC^2 each has 2 df. Let the levels of *A*, *B* and *C* be denoted by x_1, x_2 and x_3 respectively, then:

AB represents the contrasts among the response values whose x_1 and x_2 satisfy

$x_1 + x_2 = 0,1,2(mod\ 3)$,

AB^2 represents the contrasts among the response values whose x_1 and x_2 satisfy

$x_1 + 2x_2 = 0,1,2(mod\ 3)$.

AC represents the contrasts among the response values whose x_1 and x_3 satisfy

$x_1 + x_3 = 0,1,2(mod\ 3)$,

AC^2 represents the contrasts among the response values whose x_1 and x_3 satisfy

$x_1 + 2x_3 = 0,1,2(mod\ 3)$.

BC represents the contrasts among the response values whose x_2 and x_3 satisfy

$x_2 + x_3 = 0,1,2(mod\ 3)$,

BC^2 represents the contrasts among the response values whose x_1 and x_3 Satisfy

$x_2 + 2x_3$ = 0,1,2(*mod* 3).

The three-way interaction A×B×C can be partitioned into four components labelled, by *ABC*, ABC^2, AB^2C and AB^2C^2 These are the only possibilities where the first letter has exponent = 1. When the first letter has an exponent higher than one, for instance, A^2BC, to reduce it we can first square it to get $A^4B^2C^2$, and then using mod 3 arithmetic on the exponent get AB^2C^2 , i.e. a component we already have in our set. These four components partition the 8 degrees of freedom. Thus**,** A×B× C can be split up into four components denoted by *ABC*, ABC^2, AB^2C and ABC^2 Each having 2 df.

Let the levels of *A*, *B* and *C* be denoted by x_1, x_2 and x_3 respectively.

ABC, AB^2C, ABC^2 and AB^2C^2 represent the contrasts among the three groups of (x_1, x_2, x_3)

satisfying each of the four systems of equations

$x_1 + x_2 + x_3$ = 0,1,2(mod3),

$x_1 + 2x_2 + x_3$ = 0,1,2(mod3),

$x_1 + x_2 + 2x_3$ = 0,1,2(mod3),

$x_1 + 2x_2 + 2x_3$ = 0,1,2(*mod*3).

Table 3.4. Design matrix for 3^3 factorial experiments.

$x_1x_2x_3$	Run	A	B	C	AB	AB^2	AC	AC^2	BC	BC^2	ABC	ABC^2	AB^2C	AB^2C^2
0 0 0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0 0 1	2	0	0	1	0	0	1	2	1	2	1	2	1	2
0 1 0	3	0	1	0	1	2	0	0	1	1	1	1	2	2
1 0 0	4	1	0	0	1	1	1	1	0	0	1	1	1	1
0 0 2	5	0	0	2	0	0	2	1	2	1	2	1	2	1
0 2 0	6	0	2	0	2	1	0	0	2	2	2	2	1	1
2 0 0	7	2	0	0	2	2	2	2	0	0	2	2	2	2
1 1 1	8	1	1	1	2	0	2	0	2	0	0	1	1	2
1 1 0	9	1	1	0	2	0	1	1	1	1	2	2	0	0
1 0 1	10	1	0	1	1	1	2	0	1	2	2	0	2	0
0 1 1	11	0	1	1	1	2	1	2	2	0	2	0	0	1
1 1 2	12	1	1	2	2	0	0	2	0	0	1	1	2	1
1 2 1	13	1	2	1	0	2	2	0	0	1	1	2	0	1
2 1 1	14	2	1	1	0	1	0	1	2	0	1	2	2	0
2 2 0	15	2	2	0	1	0	2	2	2	2	1	1	0	0
2 0 2	16	2	0	2	2	2	1	0	2	1	1	0	1	0
0 2 2	17	0	2	2	2	1	2	1	1	0	1	0	0	2
2 2 1	18	2	2	1	1	0	0	1	0	1	2	0	1	2
2 1 2	19	2	1	2	0	1	1	0	0	2	2	1	0	2
1 2 2	20	1	2	2	0	2	0	2	1	0	2	1	1	0
0 1 2	21	0	1	2	1	2	2	1	0	2	0	2	2	0
0 2 1	22	0	2	1	2	1	1	2	0	1	0	1	2	0
1 0 2	23	1	0	2	1	1	0	2	2	1	0	2	0	2
1 2 0	24	1	2	0	0	2	1	1	2	2	0	0	2	2
2 0 1	25	2	0	1	2	2	0	1	1	2	0	1	0	1
2 1 0	26	2	1	0	0	1	2	2	1	1	0	0	1	1
2 2 2	27	2	2	2	1	0	1	0	1	0	0	2	2	1

Optimality Criteria:

According to E. A. Rady, M. M. E. Adel-Monsef, and M. M. Seyam in their publicatrion Title: Relationships among Several Optimality Criteria, defined A-,D-,E-,and G-optimality and efficiency as:

A-Optimality = $\max\limits_{xi,i=1...n}$ trace $|X'X| \equiv \min\limits_{xi,i=1....n}$ trace$(X'X)^{-1}$

D-Optimality = $\max\limits_{xi,i=1...n}$ det $|X'X| \equiv \min\limits_{xi,i=1....n}$ det$|(X'X)^{-1}|$

E-Optimality = $\max\limits_{xi,i=1...n}$ the $\min\limits_{xi,i=1...n}$ eigenvalue of$(X'X)^{-1}$

G–optimality is defined as: min max $Var\left((\hat{y})(x)\right) = \delta^2 f^T(x)\left(x^T x\right)^{-1} f(x)$

Also let ξ be a probability measure on χ then a normalized generalization relating

to $Var(\hat{y})$ is given by: $d\left(x,\xi^*\right) = f^T(x)M^{-1}(\xi)f(x)$

$$= \frac{n.\mathrm{var}\left(\hat{y}(x)\right)}{\delta^2}$$

Using this notation, we have the following definitions: ξ^* is G-optimal if and only if

$$\min_{\xi}\max_{x\in\chi} d(x,\xi) = \max_{x\in\chi} d\left(x,\xi^*\right)$$

It turns out that a sufficient condition for ξ^* to satisfy the G-optimality criterion is

$$\max_{x\in\chi} d\left(x,\xi^*\right) = P$$

Where *p* is the number of parameters in the model.

The efficiency criteria.

A-efficiency of a design ξ is defined as:

$$A(\xi) = \frac{tr\left[M^{-1}(\xi)\right]}{tr\left[M^{-1}\left(\xi_A^*\right)\right]} = \frac{P}{Ntr\left[M^{-1}\left(\xi_A^*\right)\right]}$$

Where $tr\left[M^{-1}\left(\xi_A^*\right)\right]$ is A-optimality, N is number of Rows and P is number of parameter.

D-efficiency of a design ξ is defined as:

$$D(\xi) = \left[\frac{\left|M\left(\xi_D^*\right)\right|}{M\left(\xi\right)}\right]^{1/P} = \frac{\left|M\left(\xi_D^*\right)\right|^{1/P}}{N}$$

Where $\left|M\left(\xi_D^*\right)\right|$ is D-optimality.

E-efficiency of a design ξ is defined as:

$$E(\xi) = \frac{\lambda_{\min}\left(M\left(\xi\right)\right)}{\lambda_{\min}\left(M\left(\xi_E^*\right)\right)} = \frac{P}{\lambda_{\min}\left(M\left(\xi_E^*\right)\right)}$$

Where $\lambda_{\min}\left(M\left(\xi_E^*\right)\right)$ is E-optimality.

G-efficiency of a design ξ is defined as:

$$G(\xi) = \frac{P}{\max_{x\varepsilon\chi} d(x,\xi)}$$

Where $\max_{x\varepsilon\chi} d(x,\xi)$ = N

Relative efficiency of the models

The relative efficiency of an estimator $\hat{y}_1$ to another estimator $\hat{y}_2$ is the ratio of the variance of $\hat{y}_1$ to the variance of $\hat{y}_2$

$$\text{RE } (\hat{y}_2:\hat{y}_1) = \frac{(\hat{Y}_2)}{(\hat{Y}_1)}$$

If RE ($\hat{y}_2$: $\hat{y}_1$) is greater than 1, then it means that $\hat{y}_2$ is more efficient than $\hat{y}_1$ and vice- versa.

3.4 Statistical software.

R Package was used in the cause of this research work to compute the A-,D-,E- and G-optimality criteria of the models at k =2,3.

Data analysis

This section contain the data analysis of 2^k and 3^k factorial designs for k= 2 and 3 using R Package.

Factorial design:

For k =2 that is 2^2 factorial design:

The design matrix (X) is:

X =

	[,1]	[,2]	[,3]	[,4]
[1,]	1	-1	-1	1
[2,]	1	1	-1	-1
[3,]	1	-1	1	-1
[4,]	1	1	1	1

X'X =

	[,1]	[,2]	[,3]	[,4]
[1,]	4	0	0	0
[2,]	0	4	0	0
[3,]	0	0	4	0
[4,]	0	0	0	4

Optimality.

A-optimality is maximize trace (X'X) = 16

D-optimal designs is maximizing the determinant of the information matrix (X'X) = 256

E-optimality = $\max_{xi,i=1...n}$ the $\min_{xi,i=1...n}$ eigenvalue of $X'X$ = 4

G-optimal is $\min_{\xi} \max_{x\in\chi} d\left(x,\xi\right) = \max_{x\in\chi} d\left(x,\xi^*\right) = \max_{x\in\chi} d\left(x,\xi^*\right) = p$ = 4

Efficiency.

A-efficiency $A\left(\xi\right) = 100\left(\frac{P}{Ntr\left[M^{-1}\left(\xi_A^*\right)\right]}\right) = 100\left(\frac{4}{4(16)}\right) = 6.25\,\%$

D-efficiency $D\left(\xi\right) = 100\left(\frac{\left|M\left(\xi_D^*\right)\right|^{1/P}}{N}\right) = 100\left(\frac{256^{(1/4)}}{4}\right) = 100\,\%$

E-efficiency $E(\xi) = \frac{P}{\lambda_{\min}\left(M\left(\xi_E^*\right)\right)} = 100\left(\frac{4}{4}\right) = 100\%$

G-efficiency $G(\xi) = \frac{P}{\max_{x\varepsilon\chi} d(x,\xi)} = 100\left(\frac{4}{4}\right) = 100\,\%$

Table 4.1. Optimality and efficiency of $\mathbf{2^2}$ factorial design

	A	D	E	G
Optimality	16	256	4	4
Efficiency	6.25	100%	100%	100%

For k =3 that is 2^3:

The design matrix (X) is:

X =

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]
[1,]	1	-1	-1	1	-1	1	1	-1
[2,]	1	1	-1	-1	-1	-1	1	1
[3,]	1	-1	1	-1	-1	1	-1	1
[4,]	1	1	1	1	-1	-1	-1	-1
[5,]	1	-1	-1	1	1	-1	-1	1
[6,]	1	1	-1	1	1	1	-1	-1
[7,]	1	-1	1	-1	1	-1	1	-1
[8,]	1	1	1	1	1	1	1	1

X'X =

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]
[1,]	8	0	0	0	0	0	0	0
[2,]	0	8	0	0	0	0	0	0
[3,]	0	0	8	0	0	0	0	0
[4,]	0	0	0	8	0	0	0	0
[5,]	0	0	0	0	8	0	0	0
[6,]	0	0	0	0	0	8	0	0
[7,]	0	0	0	0	0	0	8	0
[8,]	0	0	0	0	0	0	0	8

Optimality.

A-optimality is maximize trace (X'X) = sum (diag(X'X)) = 64

D-optimal designs is maximizing the determinant of the information matrix (X'X)=16777216

E-optimality = $\max\limits_{xi,i=1\ldots n}$ the $\min\limits_{xi,i=1\ldots n}$ eigenvalue of ($X'X$) = 8

G-optimal is $\min\limits_{\xi}\max\limits_{x\in\chi} d\left(x,\xi\right) = \max\limits_{x\in\chi} d\left(x,\xi^*\right) = \max\limits_{x\in\chi} d\left(x,\xi^*\right) = p$ = 8

Efficiency.

A-efficiency $A\left(\xi\right) = 100\left(\frac{P}{Ntr\left[M^{-1}\left(\xi_A^*\right)\right]}\right) = 100\left(\frac{8}{8\left(64\right)}\right) = 1.5625\,\%$

D-efficiency $= 100\left(\frac{16777216^{\left(1/8\right)}}{8}\right) = 100\,\%$

E-efficiency $E(\xi) = \frac{P}{\lambda_{\min}\left(M\left(\xi_E^*\right)\right)} = 100\left(\frac{8}{8}\right) = 100\%$

G-efficiency $G(\xi) = \dfrac{P}{\max\limits_{x \varepsilon \chi} d(x,\xi)}$ $= 100\left(\dfrac{8}{(8)}\right) = 100\,\%$

Table 4.2. Optimality and Efficiency of **2³** factorial design

	A	D	E	G
Optimality	64	16777216	8	8
Efficiency	1.56%	100%	100%	100%

The 3^k Factorial design:

For k = 2, that is 3^2

The design matrix (X) is:

X =

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]
[1,]	1	-1	1	-1	1	-1	1	-1	1
[2,]	1	0	0	-1	0	0	1	0	0
[3,]	1	1	1	-1	-1	-1	1	1	1
[4,]	1	-1	1	0	0	0	0	0	0
[5,]	1	0	0	0	0	0	0	0	0
[6,]	1	1	1	0	0	0	0	0	0
[7,]	1	-1	1	1	-1	1	1	-1	1
[8,]	1	0	0	1	0	0	1	0	0
[9,]	1	1	1	1	1	1	1	1	1

(X'X) =

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]
[1,]	9	0	6	0	0	0	6	0	4
[2,]	0	6	0	0	0	0	0	4	0
[3,]	6	0	6	0	0	0	4	0	4
[4,]	0	0	0	6	0	4	0	0	0
[5,]	0	0	0	0	4	0	0	0	0
[6,]	0	0	0	4	0	4	0	0	0
[7,]	6	0	4	0	0	0	6	0	4
[8,]	0	4	0	0	0	0	0	4	0
[9,]	4	0	4	0	0	0	4	0	4

Optimality.

A-optimality is maximize trace (X'X) = sum (diag(X'X)) = 49

D-optimal designs is maximizing the determinant of the information matrix (X'X) = 4096

E-optimality = $\max\limits_{xi, i=1\ldots n}$ the $\min\limits_{xi, i=1\ldots n}$ eigenvalue of ($X'X$) = 20.81

G-optimal is $\min\limits_{\xi} \max\limits_{x \in \chi} d\left(x,\xi\right) = \max\limits_{x \in \chi} d\left(x,\xi^*\right)$ = $\max\limits_{x \in \chi} d\left(x,\xi^*\right) = p$ = 9.00

Efficiency.

A-efficiency $A(\xi) = 100\left(\dfrac{P}{Ntr\left[M^{-1}\left(\xi_A^*\right)\right]}\right) = 100\left(\dfrac{9}{9(49)}\right) = 2.0408\,\%$

D-efficiency $D(\xi) = 100\left(\dfrac{\left|M\left(\xi_D^*\right)\right|^{1/P}}{N}\right) = 100\left(\dfrac{4096^{(1/9)}}{9}\right) = 28\,\%$

E-efficiency $E(\xi) = \dfrac{P}{\lambda_{\min}\left(M\left(\xi_E^*\right)\right)} = 100\left(\dfrac{9}{20.81}\right) = 43.25\%$

G-efficiency $G(\xi) = \dfrac{P}{\max\limits_{x\varepsilon\chi} d(x,\xi)} = 100\left(\dfrac{9}{9}\right) = 100\,\%$

Table 4.3. Optimality and E efficiency of 3^2 factorial design

	A	D	E	G
Optimality	49	4096	20.81	9
Efficiency	2.04%	28%	43.25%	100%

For k =3 that is 3^3 .

The design matrix (X) is:

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]	[,11]	[,12]	[,13]
[1,]	0	0	0	0	0	0	0	0	0	0	0	0	0
[2,]	0	0	1	0	0	1	2	1	2	1	2	1	2
[3,]	0	1	0	1	2	0	0	1	1	1	1	2	2
[4,]	1	0	0	1	1	1	1	0	0	1	1	1	1
[5,]	0	0	2	0	0	2	1	2	1	2	2	1	2
[6,]	0	2	0	2	1	0	0	2	2	2	2	1	1
[7,]	2	0	0	2	2	2	2	0	0	2	2	2	2
[8,]	1	1	1	2	0	2	0	2	0	0	1	1	2
[9,]	1	1	0	2	0	1	1	1	1	2	2	0	0
[10,]	1	0	1	1	1	2	0	1	2	2	0	2	0
[11,]	0	1	1	1	2	1	2	2	0	2	0	0	1
[12,]	1	1	2	2	0	0	2	0	0	1	1	2	1
[13,]	1	2	1	0	2	2	0	0	1	1	2	0	1
[14,]	2	1	1	0	1	0	1	2	0	1	2	2	0
[15,]	2	2	2	1	0	2	2	2	2	1	1	0	0
[16,]	2	0	0	2	2	1	0	2	1	1	0	1	0
[17,]	0	2	2	2	1	2	1	1	0	1	0	0	2
[18,]	2	2	1	1	0	0	1	0	1	2	0	1	2
[19,]	2	1	2	0	1	1	0	0	2	2	1	0	2
[20,]	1	2	2	0	2	0	2	1	0	2	1	1	0
[21,]	0	1	2	1	2	2	1	0	2	0	2	2	0
[22,]	0	2	1	2	1	1	2	0	1	0	1	2	0
[23,]	1	0	2	1	1	0	2	2	1	0	2	0	2
[24,]	1	2	0	0	2	1	1	2	2	0	0	2	2
[25,]	2	0	1	2	2	0	1	1	2	0	1	0	1
[26,]	2	1	0	0	1	2	2	1	1	0	0	1	1
[27,]	2	2	2	1	0	1	0	1	0	0	2	2	1

The information matrix (x'x) is:

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]	[,11]	[,12]	[,13]
[1,]	45	27	27	27	27	27	27	27	25	27	28	27	27
[2,]	27	45	31	27	27	27	27	27	25	27	28	28	27
[3,]	27	31	45	25	23	29	31	27	25	27	33	25	29
[4,]	27	27	25	45	27	27	27	27	23	27	29	28	27
[5,]	27	27	23	27	45	27	27	27	27	27	27	29	27
[6,]	27	27	29	27	27	45	27	27	27	27	29	27	29
[7,]	27	27	31	27	27	27	45	27	23	27	30	27	28
[8,]	27	27	27	27	27	27	27	45	27	27	29	25	29
[9,]	25	25	25	23	27	27	23	27	41	25	28	24	26
[10,]	27	27	27	27	27	27	27	27	25	45	30	25	29
[11,]	28	28	33	29	27	29	30	29	28	30	49	31	31
[12,]	27	28	25	28	29	27	27	25	24	25	31	45	27
[13,]	27	27	29	27	27	29	28	29	26	29	31	27	48

Optimality.

A-optimality is maximize trace (X'X) = sum (diag(X'X)) = 588

D-optimal designs is max. the det. of the information matrix (X'X) = 2.716634e+17

E-optimality = $\max_{xi,i=1\ldots n}$ the $\min_{xi,i=1\ldots n}$ eigenvalue of $X'X$ = 372.71

G-optimal is $\min_{\xi}\max_{x\in\chi} d\left(x,\xi\right)=\max_{x\in\chi} d\left(x,\xi^{*}\right) = \max_{x\in\chi} d\left(x,\xi^{*}\right)=p$ = 13

Efficiency.

A-efficiency $A\left(\xi\right)=100\left(\frac{P}{Ntr\left[M^{-1}\left(\xi_A^{*}\right)\right]}\right) = 100\left(\frac{13}{27\left(588\right)}\right)=0.0823\%$

D-efficiency $D\left(\xi\right)=100\left(\frac{\left|M\left(\xi_D^{*}\right)\right|^{1/P}}{N}\right) = 100\left(\frac{2.7166^{\left(1/13\right)}}{27}\right)=4\%$ 2.716634e+17

E-efficiency $E(\xi)=\frac{P}{\lambda_{\min}\left(M\left(\xi_E^{*}\right)\right)} = 100\left(\frac{13}{372.71}\right)=3.488\%$

G-efficiency $G(\xi)=\frac{P}{\max_{x\varepsilon\chi} d(x,\xi)} = 100\left(\frac{13}{27}\right)=48.148\%$

Table 4.4. Optimality and Efficiency of 3^3 factorial design

	A	D	E	G
Optimality	588	2.72	372.71	27
Efficiency	0.0823%	4%	3.488%	48.148%

Table 4.5 summary of the optimality of the models.

	Models	2^2	2^3	3^2	3^3
Optimality					
A-optimality		16	64	49	588
D-optimality		256	16777216	4096	2.72
E-optimality		4	8	9	372.71
G-optimality		4	8	9	27

Table 4.6 summary of the efficiency of the models**.**

	Models	2^2	2^3	3^2	3^3
Efficiency (%)					
A-efficiency		6.25	1.56	2.04	0.0823
D-efficiency		100	100	28	4
E-efficiency		100	100	43.28	3.488
G-efficiency		100	100	100	48.148

Relative efficiency of the models.

The relative efficiency of an estimator $\hat{y}_a$ to another estimator $\hat{y}_b$ is the ratio of the estimator $\hat{y}_a$ to the estimator $\hat{y}_b$

$$RE\ (\hat{y}_a : \hat{y}_b) = \frac{\hat{Y}_a}{\hat{Y}_b}$$

If RE ($\hat{y}_a : \hat{y}_b$) is greater than 1, then it means that $\hat{y}_a$ is more efficient than $\hat{y}_b$ and vice- versa.

Base on this research work, we take $\hat{Y}_1 = 2^2, \hat{Y}_2 = 2^3$, $\hat{Y}_3 = 3^2$ and $\hat{Y}_4 = 3^3$.

Relative efficiency between 2^k and 3^k factorial design models with respect to A-efficiency.

at k = 2.

$$RE\ \left(\hat{Y}_3 : \hat{Y}_1\right) = \frac{\hat{Y}_3}{\hat{Y}_1} = \frac{2.04}{6.25} = 0.3264\,\mathbf{.}$$

Since RE **($\hat{y}_1 : \hat{y}_3$) < 1,** then it means that $\hat{Y}_1$ is more efficient than $\hat{Y}_3$

at k = 3

RE $\left(\hat{Y}_4 : \hat{Y}_2\right) = \dfrac{\hat{Y}_4}{\hat{Y}_2} = \dfrac{0.0823}{1.56} = 0.0528$.

Since RE $\left(\hat{Y}_4 : \hat{Y}_2\right)$ **< 1,** then it means that $\hat{Y}_2$ is more efficient than $\hat{Y}_4$

Relative efficiency between 2^k^ and 3^k^ factorial design models with respect to D-Efficiency.

at k = 2.

RE $\left(\hat{Y}_3 : \hat{Y}_1\right) = \dfrac{\hat{Y}_3}{\hat{Y}_1} = \dfrac{28}{100} = 0.28$.

Since RE **($\hat{y}_1$: $\hat{y}_3$) < 1,** then it means that $\hat{Y}_1$ is more efficient than $\hat{Y}_3$

at k = 3.

RE $\left(\hat{Y}_4 : \hat{Y}_2\right) = \dfrac{\hat{Y}_4}{\hat{Y}_2} = \dfrac{4}{100} = 0.04$

Since RE $\left(\hat{Y}_4 : \hat{Y}_2\right)$ **< 1,** then it means that $\hat{Y}_2$ is more efficient than $\hat{Y}_4$

Relative efficiency between 2^k^ and 3^k^ factorial design models with respect to E-Efficiency.

at k = 2.

RE $\left(\hat{Y}_3 : \hat{Y}_1\right) = \dfrac{\hat{Y}_3}{\hat{Y}_1} = \dfrac{43.28}{100} = 0.4328$.

Since RE **($\hat{y}_1$: $\hat{y}_3$) < 1,** then it means that $\hat{Y}_1$ is more efficient than $\hat{Y}_3$

at k = 3.

RE $\left(\hat{Y}_4 : \hat{Y}_2\right) = \dfrac{\hat{Y}_4}{\hat{Y}_2} = \dfrac{3.488}{100} = 0.0349$.

Since RE $\left(\hat{Y}_4 : \hat{Y}_2\right)$ **< 1,** then it means that $\hat{Y}_2$ is more efficient than $\hat{Y}_4$

Relative efficiency between 2^k^ and 3^k^ factorial design models with respect to G-Efficiency.

at k = 2.

RE $\left(\hat{Y}_3 : \hat{Y}_1\right) = \dfrac{\hat{Y}_3}{\hat{Y}_1} = \dfrac{100}{100} = 1$

Since RE **($\hat{y}_1$: $\hat{y}_3$) = 1,** then it means that $\hat{Y}_1$ and $\hat{Y}_3$ are equal with respect to G-Efficiency.

at k = 3.

RE $\left(\hat{Y}_4 : \hat{Y}_2\right) = \frac{\hat{Y}_4}{\hat{Y}_2} = \frac{48.418}{100} = 0.4842$.

Since **RE** $\left(\hat{Y}_4 : \hat{Y}_2\right)$ **< 1,** then it means that $\hat{Y}_2$ is more efficient than $\hat{Y}_4$

Summary.

Table 3.1 shows how the designs matrix of 2^2 factorial design were constructed and table 3.2 shows how the design matrix of 2^3 factorial design were constructed. Table 3.3 shows the elements of design matrix of 3^2 factorial design and table 3.4 shows the elements of design matrix of 3^3 factorial design.

Table 4.5 present the summary of optimality of 2^k and 3^k with respect to the elements of information based criterion used that is A-, D-, E- and G-optimality

Table 4.6 present the summary of the efficiency of 2^k and 3^k with respect to the elements of information based criterion used that is A-, D-, E- and G-efficiency.

CONCLUSION.

From the analysis, we notice that 2^k factorial design is more efficient than 3^k factorial design for all values of k considered in this research (2 and 3). And when comparing the Relative Efficiency of the models (2^k and 3^k factorial designs). We noticed that 2^k is more efficient than 3^k for all value of k (2 and 3) considered in this research since the Relative Efficiency between 2^k and 3^k is less than 1.

That is R.E $\left(\frac{\hat{Y}_3}{\hat{Y}_1} \textit{ and } \frac{\hat{Y}_4}{\hat{Y}_2}\right)$ < 1 in all the elements A-, D-, E- and G- Efficiency, except for G-efficiency

Thus, from the analysis, we noticed that, the lower the number of factors in a corresponding model, the higher the efficient it becomes. Therefore, 2^k factorial designs are more efficient than 3^k factorial designs since the number of factors of 3^k designs is much larger than that of 2^k factorial design.

RECOMMENDATION

It is recommended that, for the study of factorial designs of 2^k and 3^k for k=2 and 3,

A-, D-, E-, and G-Efficiency are better when a researcher is interested in 2^k Factorial designs.

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