International Journal of Innovative Environmental Studies Research 10(4):24-35, Oct.-Dec., 2022

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A Comprehensive Numerical Simulation Model for Non-Darcy Flow including Viscous, Inertial and Convective Contributions

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ABSTRACT

In this paper, a comprehensive numerical simulation model is introduced to resolve the problem of non-Darcy flow in porous media. As represented by this model, both pressure gradient and velocity profile predicted are based on the two viscous terms of Darcy and Brinkman, Forchheimer's inertial term and Navier-Stokes' convective term. At the point of departure from the Darcian domain to the non-Darcian domain it has been found that this model predicted the dimensionless term 'Be" to be zero which agrees with Forchheimers model prediction. At 5% deviation from the Darcian flow the proposed model predicts "Be" to be 0.0756 as compared to 0.0526 predicted by Forchheimers model. The difference is due to higher flow velocity prediction by the proposed comprehensive model.

The proposed model is expected to have wide applications in the field of reservoir simulation and fluid flow in porous media in both gas and oil reservoirs. Replacing the traditional model used to predict pressure gradient at any point of space and time by the proposed model would resolve the problem of inaccurate predictions associated with non-Darcy flow. This model predicts the correct pressure gradient and flow velocity regardless of the source of deviation. The model is continuous in Darcian regime as well as non -Darcian regime.

Keywords: Simulation model, flow regime, Non-Darcy flow, Production, Pressure, Reservoir, Porous rock

INTRODUCTION

The right expression of the non-Darcy flow behaviour in porous media remains an unsolved problem. The way that many researchers and specialists look to the solution of the problem had reflected their belief and understanding of the phenomena of Non-Darcy behaviour. Some believe that the inertial term presented in Forchheimer's equation is enough to correct the extra pressure drop caused by high spatial velocity, others use Brinkman's viscous term in addition to Darcy's term to describe a solution in such circumstances.

Numerical simulation of petroleum reservoirs is aimed at predicting production so that optional development plans for the reservoir can be envisaged and the suitable production strategy can be selected. The more accurately the models represent the physics of the reservoir, the more useful the prediction made from the models can be. Most developments in the numerical reservoir simulation have revolved around accurate modeling of fluid properties and interactions and accurate representation of the storage and transmissibility properties of the porous rock material.

Reservoir simulation has been developed through a long history and has been used to address varieties of reservoir problems. However, using a conventional reservoir simulator still stand short to explain some phenomena that occur during the production process such as non-Darcy flow behaviour, fracture/matrix flow, subsidence, rock compaction, pore reduction and pore collapse, wellbore stability and sand production.

The model presented in this paper is comprehensive in nature having included not only the classic inertial term that appears in Forchheimer's equation to account for non-Darcian behaviour caused by high flow velocity, but also two viscous terms of Darcy and Brinkman and another convective term drawn from the original Navier-Stokes equation. This approach represents a general solution to all aspects of the non-linearity caused by non-Darcy behaviour; added terms drop out and diminish as the flow reaches back to the Darcian behaviour.
Our investigation into the methodologies employed for solving this class of problems was motivated by the numerous research papers and studies have been directed to this issue in an effort to generate suitable models that expresses fluid behaviour in porous media with wide spectrum of flow velocities considering both matrix and fracture contributions to fluid flow.

Models Currently in Use
Some one and a half century ago Henry Darcy1 presented an empirical model which states that the superficial velocity of fluid in a saturated porous media is related linearly to the pressure gradient within the media. Including the body force term, Darcy's law takes the following form:

$$\frac{\partial P}{\rho x} + \frac{\partial P}{\partial y} = -\frac{\mu}{K}\,U + \rho g \qquad (1)$$

Where P is the pressure, U is the volume-averaged fluid velocity, u is the fluid viscosity, K is the permeability of the porous media, p is the fluid density and g is the gravity vector. The inappropriate representation of fluid flow description of all conditions and types of flow in porous media using Darcy law has been disputed for long time. In 1901 Forchheimer[2] introduced a modified form of Darcy's equation. Basically, in addition to Darcian viscous term, an inertial term was added to account for high flow velocity:

$$\frac{\partial P}{\rho x} + \frac{\partial P}{\partial y} = -\frac{\mu}{K}\,U - \rho\beta U^2 + \rho g \qquad (2)$$

Where, β is the non-Darcy coefficient. Forchheimer's equation suffers from the difficulty associated with the evaluation of the non-Darcy coefficient β. β is mainly a function of permeability and porosity in single-phase flow cases and a function of saturations and other petrophysical parameters as well in two-phase and multi-phase cases. In some cases β presented as a function of tortuosity too, determination of tortuosity by itself adds another difficulty to the implementation of Forchheimer's equation. Li and Engler[3] articulated the correlations available for estimation of the non-Darcy coefficient for single and multiphase flow in details.
In 1997, Brinkman4 pointed out that Darcy's law does not include a term that accounts for the viscous interaction of fluid particles caused by the frictions between the molecules among themselves and with the media. Brinkman believes that Darcy's equation cannot represent boundary layers such as those which arise near an impermeable wall or a faster-moving fluid. To overcome this deficiency, Brinkman introduced his modification to Darcy's equation which is basically Darcy's equation and a viscous force term derived from the Navier-Stokes equation. Brinkman's equation takes this form:

$$\frac{\partial P}{\rho x} + \frac{\partial P}{\partial y} = -\frac{\mu}{K}\,U + \mu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \rho g \qquad (3)$$

Where, μ' is the effective viscosity which is a function of fluid viscosity and the characteristics of the porous media. μ' can be calculated using this simple equation:

$$\mu' = \frac{\mu}{\emptyset}$$

The problem with Brinkman's equation is its limit to very low volume fraction of solids, i.e. very high porosity approaches unity and the determination of the effective viscosity. Many studies49 have addressed the proper choice of effective viscosity. However, many researchers have pointed out that the Brinkman equation is not correct in situations where porosity is not close to unity which is the most common case in any underground porous medium. The term pg in eqns. 1 - 3 is too small and can be neglected.

Navier-Stokes equation1011 is very popular in addressing fluid flow between two plates and fluid flow in pipes. The simplest form of this equation in two dimensions is as following:

X-direction:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \quad (5)$$

Y-direction:

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} \quad (6)$$

U and *V* are the flow velocities in the x-direction and the y-direction, respectively. For a long time, this equation not been used one way or the other to address fluid flow in porous media. In this paper, the assumption made considers fluid flowing between two parallel plates, which can be represented by Navier-Stokes equation, and introduces solid particles filling the space between the plates. This will hinder the flow and cause flow resistance, which can be pronounced by introducing resistance terms.

Governing Equations and Simulation Technique

In this investigation, three research tasks have been set to reach the overall objective. First, selection and derivation of suitable mathematical expressions that govern the proposed type of flow scenario believed taking place in reservoirs when non-Darcy flow behaviour is encountered. Second, to numerically model the governing equations with appropriate selection of initial and boundary conditions and ensure solutions stability at acceptable degree of accuracy. Third, verify predicted results of the numerical model and conduct a parametric study to investigate its reliability and limitations.

Herein we address the issues involved with the mathematical formulations and numerical solutions of non-Darcian flow as well as its relation to the traditionally Darcian flow. The first issue which arises in the formulation of a mixed Darcian/non-Darcian flow problem is the proper mathematical form of the governing equations representing each type of flow. The second issue is the selection/derivation of suitable mathematical formulations representing this mixed system having both regions interacting and defining proper boundary and initial conditions. The flow of a single phase incompressible fluid (water or oil) in a homogeneous porous domain has been traditionally addressed by a partial differential diffusivity equation usually referred to as "diffusivity equation". The pressure drop at any point of space and time as predicted by this equation takes the following form:

$$\frac{\partial P}{\partial t} = \frac{K}{\emptyset \mu C}\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) \quad (7)$$

Where is the porosity of the porous system, C the total isothermal compressibility, and x and y are the coordinates. The pressure gradient predicted by the diffusivity equation based Forchheimer's equation12 is as following:

$$\frac{\partial P}{\partial t} = \frac{1}{C\emptyset\left(\frac{\mu}{K}+2\beta\rho v\right)}\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) - \frac{v}{\emptyset}\left(\frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\right) \quad (8)$$

Eq. 8 may represent both Darcy and non-Darcy behaviours. This equation is normally used to express gas flow. Non-Darcy representation by this equation is only pronounced in terms of the inertial effect.

The comprehensive model proposed in this paper expresses non-Darcy flow behaviour in terms of inertial, viscous and convective contributions. The pressure gradient predicted by this model includes Darcy's and Brinkman's viscous, Forchheimer's inertial and Navier-Stokes' convective terms. The derivation of the comprehensive model is based on the Navier-Stokes general equation, which accurately describes fluid flow between two parallel plates or in a pipe. Navier-Stokes equation has been modified by adding three extra resistance terms that resulted from hindering the flow and transforming the convective term to suit a porous system. Solid particles representing porous media have been assumed to fill the pipe or the space between the plates. These particles cause resistance to the fluid flow and consequently, extra pressure gradient would build up within the system.

Figure 1 is a schematic diagram of the system considered in this case. The comprehensive model predicts pressure gradient and both flow velocities in the x and y reflections at any point of space and time. The modified Navier-Stokes equation works in porous media and capable of addressing both Darcy and non-Darcy behaviours.

Predicted flow velocity in the x-direction at any time can be represented by the following equation:

$$\frac{\partial U}{\partial t} = \frac{\mu}{\rho}\frac{\partial^2 U}{\partial y^2} - \frac{\mu\emptyset}{\rho K}U - \beta\emptyset U^2 - \frac{1}{\emptyset}\left(U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right) - \frac{\emptyset}{\rho}\frac{\partial P}{\partial x} \quad (9)$$

In the y-direction, the flow velocity at any time can be predicted by this model:

$$\frac{\partial V}{\partial t} = \frac{\mu}{\rho}\frac{\partial^2 V}{\partial x^2} - \frac{\mu\emptyset}{\rho K}V - \beta\emptyset V^2 - \frac{1}{\emptyset}\left(U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y}\right) - \frac{\emptyset}{\rho}\frac{\partial P}{\partial y} \quad (10)$$

Above equations, were discretized using fully implicit formulations. Descretization of the models in Eqns. 9 and 10 is given in Appendix (A). Crank- Nicholson13"4 numerical solution method has been found the most suitable simulation technique to model this type of equations. By using this method the solution converges in less computer time, avoiding problems of instability with acceptable degree of accuracy. The flow chart of the computer program used to simulate the proposed comprehensive model is given in Appendix (B).

RESULTS AND DISCUSSION

Although the diffusivity equation derived from Darcy equation is still widely used in describing flow behaviour in oil reservoirs, critiques and questioning to its predictions abound. The effort to modify this equation continues 2'4"0'15"6 and the need to develop a better alternative become inescapable reality.

A genuine solution to express the non-Darcy flow behaviour is suggested in this paper. This solution is based on the knowledge of the sources and discrepancies that contribute to the non-linearity caused by the deviation from Darcy linear behaviour. Darcy equation emerged from an empirical approach rather than mathematical foundation. On the other hand, Navier-Stokes equation has been derived and proved mathematically. To date, it is still considered to be the corner stone of momentum transfer of fluid flow. Many types and special cases of fluid flow have been very well described using Navier-Stokes equation with some modifications. The authors believe that fluid flow in porous media with its known complexities is only a special case of Navier-Stokes fluid flow.

In this study the system described in Figure 1 is considered. Basically, we assume two parallel plates separated by a gap space where a single phase liquid (water) is supposedly flowing parallel to the plates. In this case, the basic Navier-Stokes Eqns. 5 and 6 are employed, and then the space between the two parallel plates is assumed to be filled with solid particles (representing porous media). Introducing such media would obstacle and cause extra resistance to the flow through the system and the basic Navier-Stokes equation stand short of describing such a case. The pressure gradient within the system considered in Figure 1 cannot be simply represented by the basic Navier-Stokes equation. In addition to the convective term already existed in the basic Navier-Stokes equation, the viscous terms appear in both Darcy's and Brinkman's equations and the inertial term present in Forchheimer's equation have been included in the modified Navier-Stokes models. This modification enables the Navier-Stokes model (Eqns. 9 & 10) to represent fluid flow in porous media including both Darcian and non-Darcian behaviours.

Figure 2 shows a comparison between flow velocity in the direction of flow, U, and the flow velocity perpendicular to flow direction, V. It is obvious that the vertical velocity "V' has no significant contribution to the resultant overall velocity compared to the horizontal velocity "U". Hence, the horizontal velocity is the decisive one.

Figure 3 demonstrates a comparison of predicted flow velocities in the direction of the flow, U, versus the pressure drop in the porous system as predicted by the diffusivity model of Darcy (Eq. 7), the Forchheimer model (Eq. 8) and the proposed comprehensive model (Eqns. 9 & 10). It is clear that indeed the proposed comprehensive model can represent Darcian flow in porous media by showing a linear relationship and a perfect agreement with both Darcy and Forchheimer models in the low velocity region (Darcian regime). Having included all non-Darcy terms, the comprehensive model indicates that Darcian viscous, Forchheimer's inertial and Navier-Stokes' convective terms tend to make the model predicts higher pressure gradients at the high velocity region (non-Darcian regime) while Brinkman's viscous term works in the opposite direction, pushing towards the Darcian linear trend. This conclusion can be inferred by observing the model itself (Eqns. 9 & 10). Obviously, all terms are negative quantities except the Brinkman's viscous term. This comparison also clearly indicates that depending on Forchheimer's flow equation to represent fluid flow behaviour of gas reservoirs, as currently practiced, is totally erroneous. Brinkman viscous term and Navier-Stokes convective term do not exist in Forchheimer's equation and therefore, should be included because of their important role in the non-Darcy flow behaviour region.

The dimensionless group term "Be"° which is used to distinguish between Darcian and non-Darcian flow has shown that at the point of diverging from the Darcian flow "Be" equal zero which is previously predicted by Forchheimer's model. At 5% deviation from the Darcian trend the comprehensive model predicts "Be" as following:

Equation

At the same point Forchheimer's model predicted "Be" to be 0.0526. This is due to higher velocity prediction by the comprehensive model which suggested that both velocity and pressure gradients are erroneously predicted by the traditional methods. When plotted against the pressure drop ratio predicted (Darcian to the non-Darcian), "Be" behaviour indicated perfect agreement at the Darcian region (low velocity) but, as the velocity becomes higher enough to enter the non-Darcian region, "Be" increased using both models. The comprehensive model shows even higher "Be" than Forchheimer's model. Figure 4 describes such a comparison.

CONCLUSION

The following points can be concluded from the investigation of non-Darcy modelling presented in this paper:

1. There is need to modify the model for non-Darcy fluid flow in porous media. The authors recommend that the Darcy equation, Forchheimer equation or Brinkman's equation should not be used as independent models.
2. A comprehensive model to address fluid flow in porous media has been presented in this paper. This model is capable of simultaneously representing both Darcy and non-Darcy flow in porous media.
3. This model shows that relying on Darcy and Brinkman models would lead to underestimation of pressure gradient whereas using Forchheimer model would overestimate pressure gradient.
4. Including two viscous terms, inertial term and convective term in the comprehensive model takes in consideration all sources of discrepancies that contribute to non-Darcy flow behaviour.
5. The proposed model is valid throughout Darcian and non-Darcian Flow.

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APPENDIX A

To Convert from	To	Multiply by
Ft	m	0.3048
Md	m^2	9.869E-16
psi	kPa	6.894757
Ibm/ft^3	kg/m^3	16.01845
Lbn/ft-sec	kg/m-sec	1.488 163

Nomenclature

U	Volume-Averaged flow velocity in the x-direction
V	Volume-Averaged flow velocity in the y-direction
Be	Dimensionless group
C	Isothermal compressibility factor
g	Gravitational acceleration
Φ	Porosity
K	Permeability of the porous system
P	Pressure
x	Space Coordinate in flow direction
y	Space Coordinate perpendicular to flow direction
β	Non-Darcy coefficient
t	time
μ	Viscosity of flowing fluid
μ'	Effective viscosity of flowing fluid
ρ	Fluid density

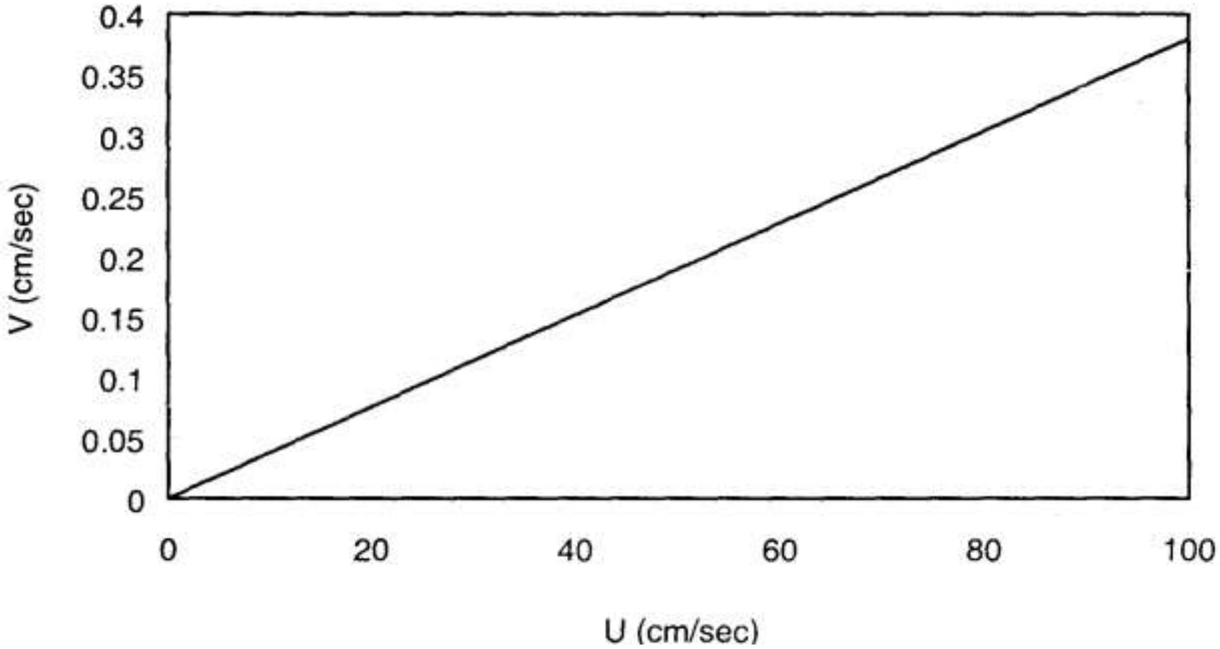


Fig. 1 Schematic of showing the boundary conditions and The numerical grid (in this study n =11 and m = 14)

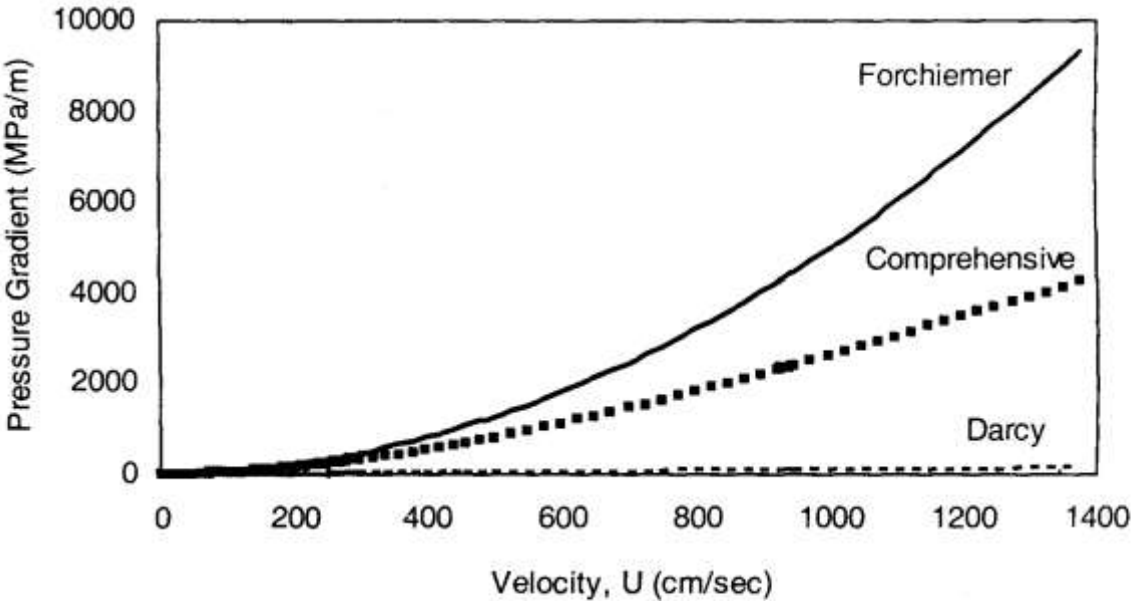


Fig. 3 Pr

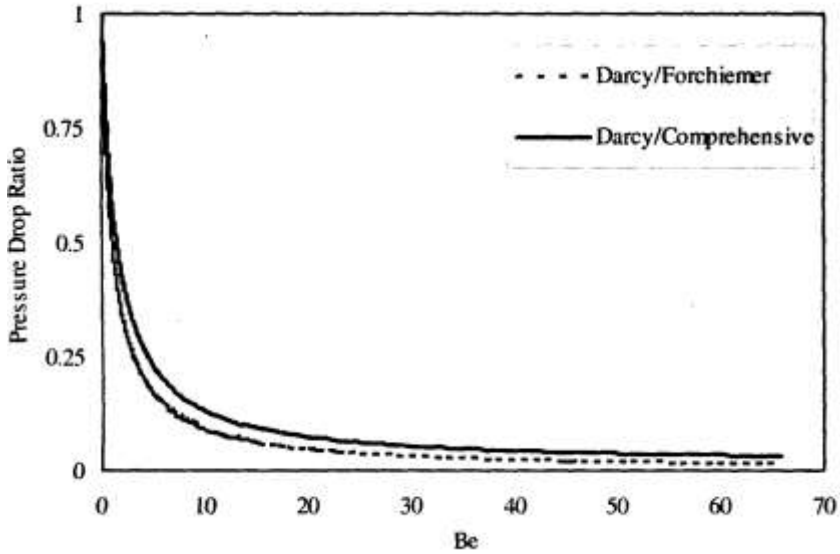


Fig. 4 Predicted gradient ratio (Darcy to Forchheimer prediction and Darcy to the comprehensive model prediction) as afunction of dimensionless number.

Appendix (A): Finite Diffrence Solution for the Comprehensive Model

The numerical solution of the transient Navier-Stokes equations adapted in this paper starts by treating the equations as an initial value problem; i.e. giving an initial state' [U(t = 0), P(t = 0)] = (U^0 , P^0), then start constructing a sequence of states (U^n, P^n) = [U(t = n), P(t = n) at predescribed times t that fulfills the continuity equation and Equations (9) and (10). To this extent, the time-derivative $\frac{\partial U}{\partial t}$ replaced by the forward difference

$$\frac{\partial U}{\partial t} = \frac{U^{n+2} - U^n}{\nabla t} \quad \text{.................................. (A-1)}$$

where ∇**t** = **t**n+1 – tn is the increment between successive time steps. Such a 'slicing' technique is typical for FD-schemes.
The time-discretized versions of continuity equation and Equations (9) and (10). can be written as:

$$0 = \nabla \cdot \mathbf{U}^{n+1} \quad \text{……………………………………. (A-2)}$$

$$\frac{U^{n+1} - U^n}{\nabla t} = \frac{\phi}{\rho}\nabla P^{n+1} - \underbrace{\frac{1}{\phi}(U^n.\nabla)\,U^n - \frac{\phi\,\mu}{\rho\,K}U^n - \phi\,\beta\,(U^2)^n + \frac{\mu}{\rho}\nabla^2 U^n}_{\psi^n} \quad \text{………………………. (A-3)}$$

Equations (A-2) and (A-3) are solved by a fractional step method: First one calculate the (physically unimportant) quantity

$$U^* = U^n + \Delta t\,\psi^n$$

Next, the new pressure field P^{n+1} is calculated by solving the Poisson equation

$$\nabla^2 P^{n+1} = \frac{\rho}{\phi\,\Delta t}\,\nabla . U^* \quad \text{……………………………………… (A-5)}$$

Taking the space-derivative of Eq. (A-5), one finds that Eq. (A-2) indeed is fulfilled with this choice of $\mathbf{P}^{n+1}$ Finally, $\mathbf{u}^{f1+1}$ is calculated from

$$u^{n+1} = u^* - \frac{\phi\quad \Delta t}{\rho}\,.\nabla\,P^{n+1} \quad \text{……………………………………… (A-6)}$$

Inserting Eq. (A-4) into Eq. (A-6) shows that $\mathbf{u}^{f1+1}$ satisfies Eq. (A-2).
In practice, Eq. (A-5) is solved by the method of *successive overrelaxation* (SOR). Introducing the notation P_m^n such that $\mathbf{P}^{(n,o)} = \mathbf{P}^n$ **;** $\mathbf{p}^{(n,\,\omega)} = \mathbf{P}^{n+1}$, the SOR-scheme reads

$$\frac{P_{m+1}^n - P_m^n}{\Im} = \nabla^2 P_m^n - \frac{\rho}{\emptyset\,\nabla t}\nabla . U^* \quad \text{………………………………………… (A-7)}$$

Where $\Im$ corresponds to the time step of the relaxation procedure. This iteration is truncated after a finite number of iterations, *m*, when sufficient convergence towards the asymptotic solution is obtained. A useful convergence criterion is

$$Max\left(\frac{|P_{m+1}^n - P_m^n|}{\Im}\right) < \varepsilon$$

Where ε is a small number. Note the completely explicit nature of the scheme, the state of the fluid at time $\mathbf{t}_{n+1}$ is calculated from the state at time $\mathbf{t}_n$ by a simple 'updating' scheme.

Space Discretization

In addition to the time-discretization, one needs to perform discretization of space. Here, the centered finite-difference operators was employed:

$$\frac{\partial g}{\partial x} = \frac{g\left(x+\frac{\Delta x}{2},y\right)-g\left(x-\frac{\Delta x}{2},y\right)}{\Delta x} \quad : \quad \frac{\partial g}{\partial y} = \frac{g\left(x,y+\frac{\Delta y}{2}\right)-g\left(x,y-\frac{\Delta y}{2}\right)}{\Delta y}$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\frac{\partial g}{\partial x}\Big|x+\frac{\Delta x}{2},y \;-\frac{\partial g}{\partial x}\Big|x-\frac{\Delta x}{2},y}{\Delta x} \quad : \quad \frac{\partial^2 g}{\partial y^2} = \frac{\frac{\partial g}{\partial y}\Big|x+\frac{\Delta y}{2} \;-\frac{\partial g}{\partial x}\Big|x,y-\frac{\Delta y}{2}}{\Delta y}$$

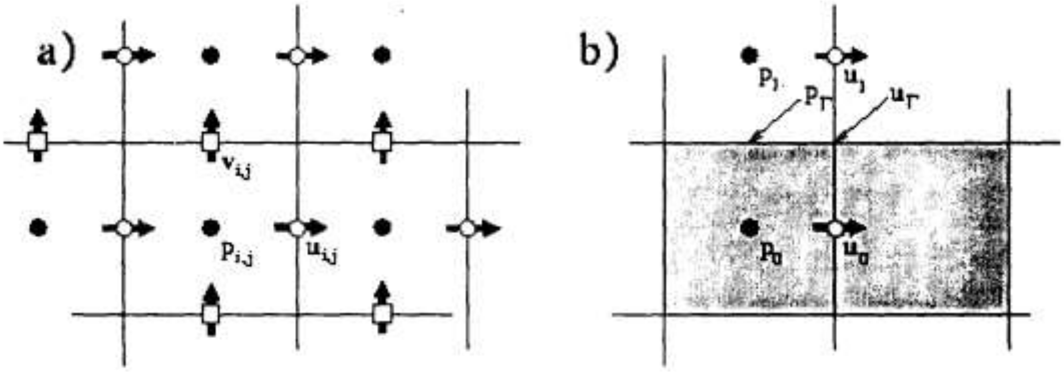
Note that this procedure imposes a lattice structure on space, where zx and y are the lattice constants in the x- and y-direction, respectively.

The values of P and $U=(u,v)^T$ are defined on a so-called staggered marker-and-cell (MAC) mesh as shown in Fig. A-1.

Boundary Conditions

We demand the fluid velocity to vanish on the fluid-solid boundary (Non-slip boundary conditions). In practice, this is done by initializing and keeping the velocity at all nodes at the fluid-solid boundary F to zero. Additional interpolations are required for some velocity nodes inside the solid, compare Fig. (A-1) shown below.

In the pressure relaxation, Eq. (A-7), a Neumann boundary condition must be employed, which is derived from projecting Eq. (A-6) on the outward normal unit vector, A, of the fluid-solid boundary F:



a) Distribution of velocity and pressure nodes on the mesh.

b) The mesh near a fluid-solid boundary. Grey sites represent solid. Some velocity and pressure nodes are indicated as in a). The values of u0 inside the solid can be interpolated by the first order formula u0=-u1 in order to mimic zero velocity at U- (non-slip boundary condition). The Neumann condition for the pressure on the fluid-solid boundary, Pr, is employed by the interpolation P0=P1.

Figure A-1: The two-dimensional staggered MAC mesh.

$$\left(\frac{\partial P}{\partial A}\right)_T^{n+1} = -\frac{\rho}{\emptyset\ \Delta t}\left(U^{n+1} - U^{*}\right)_T A_i$$

Where $(U^{n+1}$ and $U^*)$r denote the values of U^{n+1} and U^* on the boundary. Due to the non-slip boundary conditions, u^{n+1} and u^* are both zero on the boundary, and the Neumann condition reduces to an interpolation of some pressure nodes inside the solid, referred to Fig. A-1. The resulting discretized version of the Navier-Stokes equation is readily cast into an explicit time marching scheme that allows the calculation of (U^1 P^{n+1}) from (U^n , P^n).

Finally it should be noted that only unidirectional flows are considered, and a pressure gradient is applied to drive the flow. Radial flows and other more complex flows are not considered and completely omitted.

Stability and Convergence

The question of the stability of an explicit FD-scheme is very important since such schemes are only conditionally stable, and convergence of the scheme towards the physical solution is only achieved with a suitable choice of the parameters Δx , Δy, Δt and 3.

The size of the spatial discretizations Δx and Δy will usually be determined from the requirement of a sufficient spatial resolution of the geometry.

Thus, the stability criterion for the above FD-scheme reduces to the question how the involved time steps Δt and $\Im$ must be chosen. In general, stability criteria for explicit FD-schemes are obtained from the requirement that the successive corrections to the iterated quantity must decay during the iteration. This amounts to the condition that the spectral radius of the amplification matrix of the scheme be less than unity (strict Von Neumann condition). However, for our case which is SOR-scheme, the results of applying the strict Van Neumann condition leads to the following requirement.

$$\Im \leq \frac{\omega(\Delta x)^2\ (\Delta y)^2}{2((\Delta x)^2 + (\Delta y)^2)}$$

where 1< ω <2. The optimal value of ω (the one that leads to fastest convergence) depends slightly on the size of the complete lattice, but a value of ω = 1.85 is generally a good choice. It is assumed here that a Gauss-Seidel iteration for the pressure relaxation is implemented, such that updated pressure values are used as soon as they are computed.

Appendix (B) Flow Chart for the Comprehensive Model