International Journal of Innovative Mathematics, Statistics & Energy Policies
13(4):189-207, Oct-Dec., 2025

 www.seahipublications.org ISSN: 2467-852X

doi:10.5281/zenodo.17872305

Invertibility Of Univariate Time Series Model For Forecasting

[1*]Victor Edema, U.A., [2] Zebulun, G. T., & [3]Biu, E. O.

[1,2]Department of Statistics,
Ignatius Ajuru University of Education, Port Harcourt, Rivers State, Nigeria.
***Corresponding author email: uyodhu.victor-edema@iaue.edu.ng**

[3]Department of Mathematics and Statistics,
University of Port Harcourt, Harcourt, Rivers State, Nigeria.

ABSTRACT
Invertibility is a significant attribute for univariate time series models, particularly those including moving average (MA) components, such as ARMA and ARIMA models. For the model to generate accurate and understandable forecasts, it must be able to be uniquely described in terms of previous errors, which is ensured by invertibility. Multiple representations of the same data can arise from non-invertible models, which can lead to inaccurate forecasts and decreased model performance. This study introduced the needs for invertibility to resolve this inconsistent observation. For the moving average models without linear trend [MA(1), MA(2)]; the structure and parameter estimates of autoregressive infinite process were determined. Then the time series with linear trend was differenced to obtain stationarity before fitting suitable autoregressive moving average of ARIMA(0,1,1) or IMA(1,1) and ARIMA(0,1,2) or IMA(1,2). Hence, invertibility models structure and parameter estimates were established and generalized.

Keywords: Invertibility, Generalization Form, Univariate Time Series Models, Moving Average, ARIMA.

1. Introduction

Time series analysis is a powerful statistical technique which enables us to understand and predict the behavior of a variable over time. Its primary goal is to identify patterns, trends, seasonality, and other structures in the data to make predictions or gain insights into the underlying processes. It plays a pivotal role in numerous fields including finance, economics, meteorology, engineering, and healthcare, where measurements are typically taken at regular time intervals, such as daily, monthly, or yearly. It is very essential for forecasting. Accurate forecasting is crucial for businesses, economists, and researchers alike especially in today's data-driven world. Among the various forecasting approaches, univariate models, which focus on a single time series variable, have gained prominence due to their simplicity and effectiveness in capturing patterns and trends. Time-series forecasting is of great importance not only in industry, but also academia. In fact, according to Adhikari and Agrawal (2013) "a lot of efforts have been done by researchers over many years for the development of efficient models to improve the forecasting accuracy" of time-series. However, sometimes time-series forecasting methods are used without fully understanding the limitations and applicability of each method (Winters, 1960). Using forecasts from a model that does not apply to the given time-series may lead to inaccurate predictions. Hence, a forecaster has to be careful in choosing a forecasting model suitable to the situation at hand (Purthan et al., 2014). Forecasting models aim to capture and predict future patterns, enabling informed decision-making and effective planning. This research invented some models that can aid forecasting accuracy using invertibility. Invertibility refers to the capacity of a model to reconstruct the past observations based on the current and previous forecasts. It ensures that the model's estimated parameters are reliable and align with the true data generating process, leading to accurate and trustworthy predictions. Invertibility traces its roots to the development of autoregressive (AR) and moving average (MA) models. Theoretical foundations for these models were established in the early 20th century by statisticians and mathematicians such as Norbert Wiener, Udny Yule, and R.A. Fisher. These models provided a framework for analyzing and forecasting time series data by capturing the relationships between current and past observations. Autoregressive models were introduced in the early 20th century by the mathematician Norbert Wiener and further developed by the statistician Yule and others. In the early stages, the focus was primarily on the theoretical properties and applications of AR models. Moving average (MA) models were introduced in the mid-20th century as an alternative approach to time series modelling. The concept of invertibility became prominent with MA models, as it was crucial for ensuring the stability and meaningful interpretation of the models. The pioneering work on MA models was done by Peter Whittle and George Box, among others. The development of the Box-Jenkins methodology in the 1960s and 1970s played a significant role in popularizing the concept of invertibility in time series analysis. The Box-Jenkins approach emphasized the importance of diagnostic checks, including checking for invertibility, to ensure the validity of Autoregressive Integrated Moving Average (ARIMA) models. The methodology provided a systematic framework for model identification, estimation, and diagnostic checking. In the subsequent years, researchers made significant theoretical advancements in understanding invertibility. The mathematician George P. H. Styan introduced the notion of a causal autoregressive moving average (ARMA) process, which highlighted the importance of invertibility for causal interpretations. Researchers also explored the mathematical properties and conditions for invertibility in AR and MA models. Over time, the concept of invertibility has been extended to more complex time series models, such as multivariate models and state space models. Invertibility continues to be a fundamental consideration in various time series applications, including economic forecasting, financial modelling, and signal processing. The historical development of invertibility reflects the ongoing efforts of statisticians and econometricians to improve the accuracy, interpretability, and reliability of time series models. The concept has evolved hand in hand with the development of new methodologies, theoretical advancements, and practical applications in the field of time series analysis.

The previous work of Arimie et al (2018) on a medical data used for forecasting obtained from MA model showed that the forecast was not stochastic in nature but constant. This was due to the nature of data (small sample size) used. Sample size significantly impacts forecasting accuracy in univariate time series models. This is not different in medical data. There is therefore the need for model invertibility. Inverting a univariate model ensures that the forecasted values align with the original scale of the data. Ordinarily, a larger sample size generally facilitates more accurate estimation of model parameters. Because with more observations, the model can estimate the parameters more reliably and generate more accurate forecasts. The increased data points allow for better capturing of seasonality, long-term trends, and other relevant patterns, leading to improved forecasting performance. Again, working with transformed data in the case of non-stationary data, the forecasts obtained from the model may not be directly interpretable or comparable to the original data. Inverting the model allows for forecasts in the original scale, making them more meaningful and useful for decision-making, hence this study on the generalization form of the univariate invertibility time series models where Moving Average (MA) models were inverted to an infinite model to obtain a stochastic (random) forecast model. This was achieved by obtaining the invertibility model for the selected models MA(1); MA(2); IMA(1,1) or ARIMA(0,1,1); ARIMA(0,1,2) or IMA(1,2), and generalizing the obtained invertibility models parameters of the selected MA(1), MA(2), IMA(1,1), IMA(1,2) models. The study will help to improve forecasting precisions, contribute to enhancing the reliability and precision of forecasts generated from univariate time series models, and contribute to the advancement of time series forecasting techniques.

In practice, when estimating an ARIMA model, it is common to impose invertibility constraints on the parameters to ensure that the model is invertible. This can be done by setting appropriate bounds or constraints on the MA parameters during the estimation process. Ensuring invertibility during the model-building process is crucial for obtaining stable and accurate time series models. Various techniques can be employed to achieve this, depending on the type of model being used (Box et. al, 2008; Box et al. 2015; Tibshirani 1996; Hamilton 1994; Lütkepohl 2005; Hyndman & Athanasopoulos 2018; Brockwell & Davis 2016.

Precise predictions for medical data are essential for planning and allocating resources. Models invertibility guarantees the accuracy of the forecasts. This enables healthcare professionals to make defensible choices (Hyndman & Athanasopoulos, 2018); guarantees that the stochastic process that represents a financial time series may be appropriately described (Hamilton 1994); that models are robust and able to handle the non-stationary nature of environmental data in environmental studies (Shumway & Stoffer 2017). Also, invertibility in demand forecasting models enables businesses to keep a balance between supply and demand, lowering the expenses related to overstocking or understocking, (Box et al. 2015)

Granger and Allan (1978) proposed a generalized definition of invertibility and applied it to linear, non-linear and bilinear models. Using this definition, some intuitively attractive non-linear moving average models are shown to be never invertible. However, bilinear models are found to be both non-linear generalizations of ARMA models and to be sometimes invertible. They also found that some invertible bilinear models do produce superior forecasts compared to linear models having the same autocovariances. Invertibility is a crucial property of ARMA or ARIMA models that ensure stability, interpretability, and reliable forecasting. It allows for a meaningful analysis of the model's components and facilitates the use of standard statistical techniques such as maximum likelihood estimation, making the model more practical for time series analysis and forecasting. Without invertibility, parameter estimation can become unstable, leading to unreliable predictions (Box et al. 2015).

Invertibility has also been studied recently in relation to nonlinear and non-stationary time series models. The invertibility of nonlinear time series models, such as smooth transition autoregressive (STAR) and threshold autoregressive (TAR) models, was examined by Chan and Palma (2015). According to their findings, the intricate interactions among the nonlinear components of these models make ensuring invertibility more difficult. However, the authors suggest cutting-edge techniques for ensuring and verifying invertibility, which can greatly improve these models' accuracy and robustness.

2. MATERIALS AND METHODS

Autoregressive moving average (ARMA) model

The ARMA (Autoregressive Moving Average) model is a popular class of univariate time series models that combines autoregressive (AR) components

$$X_t = \emptyset_o + \emptyset_1 X_{t-1} + \emptyset_2 X_{t-2} + \dots + \emptyset_p X_{t-p} + e_t = \emptyset_0 + \sum_{i=1}^{p} \emptyset_i X_{t-i} + e_t \quad (2.1)$$

and moving average (MA) components

$$X_t = \mu + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \dots + \theta_q e_{t-q} = \mu + \sum_{j=1}^{q} \theta_j e_{t-j} + e_t \quad (2.2)$$

to capture both the AR and MA effects in the time series. The model parameters ($\emptyset_1, \emptyset_2, \dots, \emptyset_p$) from AR models and $(\theta_1, \theta_2, \dots, \theta_q)$ from the MA models are estimated from the data.

Autoregressive Integrated Moving Average (ARIMA)

ARIMA (Autoregressive Integrated Moving Average) is an extension of ARMA. It combines autoregressive (AR), differencing (I), and moving average (MA) components. The model is also widely used for analyzing and predicting time series data.

Mathematically, an ARMA or ARIMA model is invertible if the polynomial representation of the moving average (MA) component has all its roots outside the unit circle. Roots outside the unit circle guarantee that the model does not exhibit any explosive behaviour (that is, the model's values increase exponentially over time, leading to unstable and unbounded growth) and can be represented as an infinite autoregressive (AR) model.

A notable property of time series model is that the model's parameters can be uniquely estimated based on the observed data. In other words, an invertible model can be transformed into a model with a unique representation.

Consider the univariate class of time series models which take the form

$$x_t = f\left(x_{t-j}, e_{t-j}, \quad j = 1, \dots, P\right) + e_t \qquad (2.3)$$

where e_t is pure white noise and is an unobserved input to the system. If this type of model is to be used for forecasting, and provided the function $f(\ )$ does contain some $e_{t-j}, j \ \geq 1$, it is necessary to be able to estimate the e_t sequence from the observed x_t's. A natural way of doing this is to assume values for as many initial, or starting up $e's$ as are necessary, i.e. $\hat{e}_{-j}, j = 0, \dots, (P-1)$ say. Assuming now that $x_{-j}, j = 0, \dots, (P-1)$ are also observed, then e_1 can be estimated directly from the assumed known generating formula (l.1) by

$$\hat{e}_1 = x_1 - f\left(x_{1-j}, \hat{e}_{1-j}, j = 1, \dots, p\right) \qquad (2.4)$$

and then $\hat{e}_2$ is estimated by

$$\hat{e}_2 = x_2 - f\left(x_{2-j}, \hat{e}_{2-j}, j = 1, \dots, p\right) \qquad (2.5)$$

and so forth. Thus, using the assumed starting up values an iterative procedure can be used to estimate all $\hat{e}_t$ based on the observed $x's$ and the generating model. The error that will almost certainly 'arise from such a procedure will be denoted

$$e_t = \ e_t - \hat{e}_t$$

We shall say that the model (1.2) is invertible if

$$E[e_t^2] \to 0 \ as\ t \to \infty, \qquad (2.6)$$

which assumes that the correct generating formula is known completely, both in form and the actual parameter values.

$$\hat{e}_t = \ x_t - g\left(x_{t-k}, k = 1, \cdots, t + P, \hat{e}_{t-j,} \ j = 1, \cdots, P\right) \qquad (2.7)$$

where $\{\hat{e}_{1-j}\}$are given estimates of the appropriate random variables. The condition (2.6) therefore implies that a model is invertible if the variance of the error involved in estimating e_t in terms of a finite number of past and present $x's$ tends to zero as that number tends to infinity, unconditional on the starting up values.

If the parameter values in the model are not known exactly, as it would occur if they were estimated from an earlier finite length of data, for instance, (2.6) might be replaced by

$$E[e_t^2] \to c \quad as \quad t \to \infty, \qquad (2.8)$$

where $c < \infty$ is some constant (Granger & Allan, 1978).

The term "invertibility" is often used in relation to autoregressive moving average (ARMA) models or their extensions like autoregressive integrated moving average (ARIMA) models. These models are widely used for forecasting and analyzing time series data. According to Box and Jenkins (1970), an ARMA(p,q) model is said to be invertible if it can be represented as an infinite order MA model, i.e., if the roots of the polynomial $(1 - \theta_1 B - \theta_2 B^2 - \dots - \theta_q B^q)$ are all outside the unit circle, where B is the backward shift operator.

For an ARMA model to be invertible, certain conditions need to be satisfied. In particular:

AR component: The autoregressive coefficients should be within the unit circle (while the roots are outside the unit circle). This ensures that the impact of past observations on the current observation diminishes over time. If the coefficients are outside the unit circle, the model may explode or exhibit explosive behaviour.

MA component: The moving average coefficients should also be within the unit circle. This ensures that the model is well-behaved and converges to a unique representation.

If a model is not invertible, it may lead to estimation problems and unreliable predictions.

The invertibility of an ARIMA (Autoregressive Integrated Moving Average) model is related to the moving average (MA) component of the model. An ARIMA model is specified by three components: autoregressive (AR), differencing (I), and moving average (MA).

The invertibility property of an ARIMA model ensures that the model can be transformed into an equivalent representation without the MA component. In other words, if an ARIMA model is invertible, the past values of the observed series can be used to predict future values without needing the MA terms. That is, converting the MA component to AR(∞).

An ARIMA(p, d, q) model can be represented as:

$$(1 - \emptyset_1 B - \emptyset_2 B^2 - \dots - \emptyset_p B^p)(1 - B)^d X_t = (1 + \theta_1 B + \theta_2 B^2 + \dots + \theta_q B^q)e_t \qquad (2.9)$$

where X_t is the observed series, ε_t is a sequence of white noise residuals, and B is the backward shift, $\emptyset$ and θ are the autoregressive and moving average parameters, respectively. The differencing parameter is denoted by d.

To determine the invertibility of the ARIMA model, we examine the roots of the AR and MA characteristic polynomials. The AR characteristic polynomial is represented by

$$1 - \emptyset_1 B - \emptyset_2 B^2 - \dots - \emptyset_p B^p, \qquad (2.10)$$

replacing the backward shift B by a variable (z) and the MA characteristic polynomial is represented by

$$(1 + \theta_1 z + \theta_2 z^2 + \ldots + \theta_q z^q), \tag{2.11}$$

where z is a complex number.

Models invertibility is established using backward shift notation and repeated substitution until the parameters at which the model was adequate for invertibility are obtained.

Differencing: Non-stationary series can be made stationary by non-seasonal differencing to remove the trend. The backward shift operator, B, is defined as

$$B^m X_t = X_{t-m} \tag{2.1}$$

The backward difference operator, $\nabla = 1 - \mathbf{B}$, is defined by

$$\nabla^m X_t = (1 - B)^m X_t \tag{2.2}$$

For a series differenced "d" times, the stationary series Y_t is defined as

$$Y_t = \nabla^d X_t = (1 - B)^d X_t \tag{2.3}$$

and

$$Z_t = \nabla_s X_t = X_t - X_{t-s} = (1 - B^s) X_t \tag{2.4}$$

for a series with seasonal period s. $\nabla_s = 1 - B^S$.

Occasionally,

$$W_t = (1 - B)_s^d (1 - B^S)^D X_t = \nabla^d \nabla_S^D X_t \tag{2.5}$$

is used when the series has both trend and seasonal component.

2.1 Model Specification

2.1.1. Moving Average MA(q) model

MA(q) model is given by:

$$Y_t = \mu + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \ldots + \theta q e_{t-q} \tag{2.6}$$

$$y_t = (1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q) e_t$$

Generally,

$$Y_t = \mu + \sum_{j=1}^{q} \theta_q e_{t-q} + e_t \tag{2.7}$$

where

$$e_t = \left(\sum_{j=0}^{\infty} w_j B^j\right) y_t$$

Case 1: Order 1

MA(1) or ARMA(0,1): $y_t = \theta_1 e_{t-1} + e_t$

or

$$y_t = (1 + \theta_1 B) e_t \tag{2.8}$$

Case 2: Order 2

MA(2) or ARMA(0,2)]: $y_t = \theta_1 e_{t-1} + \theta_2 e_{t-2} + e_t$ (2.9)

or

$$y_t = (\theta_1 B + \theta_2 B^2 + 1) e_t \tag{2.10}$$

AR Representation: If the polynomial $\theta(B)$ has roots outside the unit circle, the MA(q) process can be represented as:

$$e_t = \sum_{i=0}^{\infty} \emptyset_i Y_{t-i} \tag{2.11}$$

where $\emptyset_i$ are coefficients that decay sufficiently fast for series convergence.

Series Convergence: The condition that the absolute value of the roots is greater than 1 (that is, $|z| > 1$). If the series converges, that is, $|w_i| < 1$ then this infinite models series converges to a finite value, hence MA(q) processes are called invertible, which implies that the MA process can be in AR representation.

Invertibility of MA(q) to AR(∞) [Infinite-order autoregressive process, AR(∞)]

Any stationary MA(q) process can be expressed as an AR(∞) process. Suppose we have an MA(1) process with $\mu = 0$.

$$y_t = \theta_1 e_{t-1} + e_t \tag{2.12}$$

thus

$$e_t = y_t - \theta_1 e_{t-1} = y_t - \theta_1(y_{t-1} - \theta_1 e_{t-2}) = y_t - \theta_1 y_{t-1} + \theta_1^2 e_{t-2} \tag{2.13}$$

For n steps we have

$$e_t = y_t - \theta_1 y_{t-1} + \theta_1^2 y_{t-2} - \theta_1^3 y_{t-3} + \cdots + (-\theta_1)^n y_{t-n} + (-\theta_1)^{n+1} e_{t-n-1}$$

$$= \sum_{j=0}^{\infty} (-\theta_1)^j y_{t-j} + (-\theta_1)^{n+1} e_{t-n-1} \tag{2.14}$$

thus resulting in

$$e_t = y_t - \theta_1 y_{t-1} + \theta_1^2 y_{t-2} - \theta_1^3 y_{t-3} + \cdots = \sum_{j=0}^{\infty} (-\theta_1)^j y_{t-j}$$

equivalently

$$y_t = \theta_1 y_{t-1} - \theta_1^2 y_{t-2} + \theta_1^3 y_{t-3} + \cdots + e_t = \sum_{j=1}^{\infty} (-\theta_1)^j y_{t-j} + e_t \qquad (2.15)$$

If $|\theta_1| < 1$ then this infinite series converges to a finite value. Such MA(q) processes are invertible**.**

Invertibility of AR(p) to MA(∞) [Infinite Moving Average Processes, MA(∞)]

An infinite-order moving average process**, MA (∞)**

$$y_i = \mu + e_i + \sum_{j=1}^{\infty} \emptyset_j e_{i-j} \qquad (2.16)$$

where the following infinite series is finite (i.e. converges to a real value)

$$\sum_{j=1}^{\infty} |\emptyset_j| < \infty \qquad (2.17)$$

and

$$e_t \sim N(0,\sigma) \qquad cov(e_i, e_j) = 0 \ if \ i \neq j \qquad (2.18)$$

Then, MA(∞) process is express as

$$y_i = \mu + \sum_{j=0}^{\infty} \emptyset_j e_{i-j} \qquad (2.19)$$

where it is assumed that $\psi_0 = 1$.

Note: Invertibility holds if and only if the roots of the MA characteristic equation $\theta(\beta) = 0$ are all outside the unit circle for the ARIMA (1,0,1) Model. A finite order MA is an infinite order AR and any finite order AR is an infinite order MA**.** If the process is stationary and invertible, then the model can be inverted to obtain the forecast.

2.1.2 Autoregressive integrated moving average ARIMA (p, d, q) model

An autoregressive time series model of order p, denoted by AR(p), may be represented as

$$Y_t = \emptyset_0 + \sum_{i=1}^{p} \emptyset_p Y_{t-p} + e_t \qquad (2.20)$$

where ϕ_0 is a constant, e_t , $t \in Z$ is a zero-mean white noise process, and Y_t is stationary. Model (2.20) is used to predict the present value of Y_t, considering the past values up to order p of the functions $Y_{t-1}, Y_{t-2}, \cdots, Y_{t-p}$.

A Moving Average process of order q, denoted by MA(q), is given as

$$Y_t = \mu + \sum_{j=1}^{q} \theta_q e_{t-q} + e_t \tag{2.21}$$

where μ is a constant, e_t , $t \in Z$ is a white noise process. (Robert & David, 2011; Bowerman & Connell, 2007; Chatfield, 2004; Pankratz, 1983).

The Autoregressive and Moving Average models denoted by ARMA (p, q) is represented as

$$Y_t = \lambda + \sum_{i=1}^{p} \emptyset_i Y_{t-i} + \sum_{j=1}^{q} \theta_j e_{t-j} + e_t \tag{2.22}$$

where λ is a constant, e_t, t ∈ Z is a zero mean white noise process. autoregressive integrated moving average process denoted by ARIMA (p, d, q) is represented mathematically by

$$\nabla^d Y_t = \delta + \nabla^d Y_{t-1} + \dots\dots\dots + \nabla^d Y_{t-p} + \theta_1 e_{t-1} + \dots\dots\dots\ \theta_q e_{t-q} + e_t \tag{2.23}$$

where ∇^d is the d^{th} difference of the time series, δ is a constant and e_t, t ∈ Z is a zero mean white noise process (Box et al. 1994); Chatfield, 2004).

ARIMA(0,1,1) or IMA (1,1) model:

$$y_t = y_{t-1} + \theta_1 e_{t-1} + e_t \tag{2.24a}$$

$$\nabla y_t = \theta_1 e_{t-1} + e_t$$

$$(1-B)y_t = (\theta_1 B + 1)\, e_t$$

where

$$e_t = \left(\sum_{j=0}^{\infty} w_j B^j\right) y_t \tag{2.24b}$$

ARIMA(0,1,2) or IMA(1,2)

$$y_t = y_{t-1} + \theta_1 e_{t-1} + \theta_2 e_{t-2} + e_t \tag{2.25}$$

$$\nabla y_t = \theta_1 e_{t-1} + \theta_2 e_{t-2} + e_t$$

$$(1-B)y_t = (\theta_1 B + \theta_2 B^2 + 1)\, e_t$$

ARIMA Representation (Finite to Infinite Model)

Invertibility Condition:

The representation of ARIMA model to an infinite model "AR(∞)" or AR representation is defined as

$$\mathrm{X}_t = \sum_{i-1}^{\infty} \varphi_i \mathrm{X}_{t-i} + \varepsilon_t$$

or

$$\mathrm{X}_t - \sum_{i-1}^{\infty} \varphi_i \mathrm{X}_{t-i} = \left(1 - \sum_{i-1}^{\infty} \varphi_i \mathrm{B}^i\right) \mathrm{X}_t = \varepsilon_t$$

(Box et al. 2008). (2.26)

Furthermore, the representation of an ARIMA model to infinite a model "MA(∞)" or MA is defined as

$$X_t = \sum_{i-1}^{\infty} \omega_i \varepsilon_{t-i} + \varepsilon_t$$

or

$$X_t = \sum_{i-0}^{\infty} \omega_i \varepsilon_{t-i} = \left(\sum_{i-0}^{\infty} \omega_i B^i \right) \varepsilon_t \;\; ; \omega_0 = 1$$

3. RESULTS

The invertibility model for the selected models MA(1); MA(2); IMA(1,1) or ARIMA(0,1,1); ARIMA(0,1,2) or IMA(1,2) were obtained, and the models' parameters generalized.

3.1 Invertibility model for MA(1) [to AR(∞) model]

MA (1) is given by:

$$y_t = \theta_1 e_{t-1} + e_t$$

$$y_t = (1 + \theta_1 B) e_t \tag{3.1}$$

Recall equation (2.24b)

$$e_t = \left(\sum_{j=0}^{\infty} w_j B^j \right) y_t$$

$$= (w_0 B^0 + w_1 B + w_2 B^2 + w_3 B^3 + \cdots) y_t$$

Substituting for e_t in equation (4.1), we have

$$y_t \quad = (1 + \theta_1 B)\left(\textstyle\sum_{j=0}^{\infty} w_j B^j\right) y_t$$

That is,

$$y_t \quad = (1 + \theta_1 B)(w_0 B^0 + w_1 B + w_2 B^2 + w_3 B^3 + \cdots) y_t$$

$$1 = (1 + \theta_1 B)(1 + w_1 B + w_2 B^2 + w_3 B^3 + \cdots)$$

$$1 = 1 + w_1 B^1 + w_2 B^2 + w_3 B^3 + \cdots + \theta_1 B + \theta_1 w_1 B^1 + \theta_1 w_2 B^2 + \theta_1 w_3 B^3 + \cdots$$

Taking coefficient

B^1: $\quad 0 = w_1 + \theta_1$

$$w_1 = -\theta_1$$

B^2: $\quad 0 = w_2 + \theta_1 w_1$

$$w_2 = -\theta_1 w_1$$

$$w_2 = -\theta_1(-\theta_1) = \theta_1{}^2$$

B^3: $\quad 0 = w_3 + \theta_1 w_2$

$$w_3 = -\theta_1 w_2$$

$$w_3 = -\theta_1(\theta_1{}^2) = -\theta_1{}^3$$

B^4: $\quad 0 = w_4 + \theta_1 w_3$

$$w_4 = -\theta_1 w_3$$

$$w_4 = -\theta_1(-\theta_1{}^3) = \theta_1{}^4$$

… etc

The parameter estimates from MA(1) to AR(∞): Infinite order of autoregressive process is $-\theta_1, \theta_1{}^2, -\theta_1{}^3, \theta_1{}^4, -\theta_1{}^5, \ldots$ for $w_1, w_2, w_3, w_4, w_5, \ldots$ respectively.

The invertibility stationary model for AR(∞) process is given by:

$$y_t + w_1 y_{t-1} + w_2 y_{t-2} + w_3 y_{t-3} + \cdots = e_t$$

or

$$y_t = -w_1 y_{t-1} - w_2 y_{t-2} - w_3 y_{t-3} - \cdots - e_t$$

Therefore, MA(1) to AR(∞) model is:

$$y_t - \theta_1 y_{t-1} + \theta_1{}^2 y_{t-2} - \theta_1{}^3 y_{t-3} + \theta_1{}^4 y_{t-4} \ldots = e_t$$

or

$$y_t = \theta_1 y_{t-1} - \theta_1{}^2 y_{t-2} - \theta_1{}^3 y_{t-3} - \theta_1{}^4 y_{t-4} + \cdots + e_t$$

Generally,

$$\textstyle\sum_{j=0}^{\infty} (-\theta_1)^j y_{t-j} = e_t$$

or

$$y_t = \sum_{j=0}^{\infty} (-1)^{j+1} \theta_1{}^j y_{t-j} + e_t$$

3.2 Invertibility model for MA(2) [to AR(∞) model]

MA (2) is given by:

$$y_t = \theta_1 e_{t-1} + \theta_2 e_{t-2} + e_t$$

$$y_t = (\theta_1 B + \theta_2 B^2 + 1) e_t$$

But

$$e_t = \left(\sum_{j=0}^{\infty} w_j B^j\right) y_t$$

$$= (w_0 B^0 + w_1 B + w_2 B^2 + w_3 B^3 + \cdots) y_t$$

Substituting for e_t we have,

$$y_t = (1 + \theta_1 B + \theta_2 B^2)\left(\sum_{j=0}^{\infty} w_j B^j\right) y_t$$

$$1 = (1 + \theta_1 B + \theta_2 B^2)(w_0 B^0 + w_1 B^1 + w_2 B^2 + w_3 B^3 + \cdots)$$

$$1 = (1 + \theta_1 B + \theta_2 B^2)(1 + w_1 B^1 + w_2 B^2 + w_3 B^3 + \cdots)$$

$$1 = 1 + w_1 B^1 + w_2 B^2 + w_3 B^3 + \cdots + \theta_1 B + \theta_1 w_1 B^2 + \theta_1 w_2 B^3 + \theta_1 w_3 B^4 + \cdots + \theta_2 B^2 + \theta_2 w_1 B^3 + \theta_2 w_2 B^4 + \theta_2 w_3 B^5 + \cdots$$

Taking coefficient

B^1: $\quad 0 = w_1 + \theta_1$

$$w_1 = -\theta_1$$

B^2: $\quad 0 = w_2 + \theta_1 w_1 + \theta_2$

$$w_2 = -\theta_1 w_1 - \theta_2$$

$$w_2 = (-\theta_1)(-\theta_1) - \theta_2$$

$$w_2 = \theta_1^{\,2} - \theta_2$$

B^3: $\quad 0 = w_3 + \theta_1 w_2 + \theta_2 w_1$

$$w_3 = -\theta_1 w_2 - \theta_2 w_1$$

$$w_3 = -\theta_1(\theta_1^{\,2} - \theta_2) - \theta_2(-\theta_1)$$

$$w_3 = -\theta_1^{\,3} + \theta_1\theta_2 + \theta_1\theta_2$$

$$w_3 = -\theta_1^{\,3} + 2\theta_1\theta_2$$

OR

$$w_3 = -\theta_1(\theta_1^{\,2} - 2\theta_2)$$

B^4: $\quad 0 = w_4 + \theta_1 w_3 + \theta_2 w_2$

$$w_4 = -\theta_1 w_3 - \theta_2 w_2$$

$$w_4 = -\theta_1(-\theta_1^{\,3} + 2\theta_1\theta_2) - \theta_2(\theta_1^{\,2} - \theta_2)$$

$$w_4 = \theta_1^{\,4} - 2\theta_1^{\,2}\theta_2 - \theta_1^{\,2}\theta_2 + \theta_2^{\,2}$$

$$w_4 = \theta_1^{\,4} - 3\theta_1^{\,2}\theta_2 + \theta_2^{\,2}$$

or

$$w_4 = \left(\theta_1{}^2 - \theta_2\right)^2 - \theta_1{}^2\theta_2$$

B^5: $\quad 0 = w_5 + \theta_1 w_4 + \theta_2 w_3$

$$w_5 = -\theta_1 w_4 - \theta_2 w_3$$

$$w_5 = -\theta_1\left(\theta_1{}^4 - 3\theta_1{}^2\theta_2 + \theta_2{}^2\right) - \theta_2\left(-\theta_1{}^3 + 2\theta_1\theta_2\right)$$

$$w_5 = -\theta_1{}^5 + 3\theta_1{}^3\theta_2 - 3\theta_1\theta_2{}^2 + \theta_1{}^3\theta_2$$

$$w_5 = -\theta_1{}^5 + 3\theta_1{}^3\theta_2 + \theta_1{}^3\theta_2 - 3\theta_1\theta_2{}^2$$

$$w_5 = -\theta_1{}^5 + 4\theta_1{}^3\theta_2 - 3\theta_1\theta_2{}^2$$

Generally,

$$w_k = -\theta_1 w_{k-1} - \theta_2 w_{k-2} \qquad \forall k > 1$$

3.3 Invertibility model for IMA(1,1) or First difference of MA(1) [to AR(∞) model]

$$y_t = y_{t-1} + \theta_1 e_{t-1} + e_t \tag{3.3}$$

$\nabla y_t = \theta_1 e_{t-1} + e_t$

$(1 - B)y_t = (\theta_1 B + 1)\, e_t$

Recall again, equation (2.24a)

$$e_t = \left(\sum_{j=0}^{\infty} w_j B^j\right) y_t$$

$$(1 - B)y_t = (\theta_1 B + 1)\left(\sum_{j=0}^{\infty} w_j B^j\right) y_t$$

$$(1 - B) = (1 + \theta_1 B)\,(1 + w_1 B^1 + w_2 B^2 + w_3 B^3 + w_4 B^4 + \cdots)$$

$$1 - B = 1 + w_1 B^1 + w_2 B^2 + w_3 B^3 + w_4 B^4 + \cdots + \theta_1 B + \theta_1 w_1 B^2 + \theta_1 w_2 B^3 + \theta_1 w_3 B^4 + ..$$

Taking Coefficient

B^1: $\quad -1 = w_1 + \theta_1$

$$w_1 = -(1 + \theta_1)$$

B^2: $\quad 0 = w_2 + \theta_1 w_1$

$$w_2 = -\theta_1 w_1$$

$$w_2 = -\theta_1[-(1 + \theta_1)]$$

$$w_2 = \theta_1(1 + \theta_1)$$

B^3: $\quad 0 = w_3 + \theta_1 w_2$

$$w_3 = -\theta_1 w_2$$

$$w_3 = -\theta_1[\theta_1(1 + \theta_1)]$$

$$w_3 = -\theta_1{}^2(1+\theta_1)$$

B^4: $\quad 0 = w_4 + \theta_1 w_3$

$$w_4 = -\theta_1 w_3$$

$$w_4 = -\theta_1[-\theta_1{}^2(1+\theta_1)]$$

$$w_4 = \theta_1{}^3(1+\theta_1)$$

B^5: $\quad 0 = w_5 + \theta_1 w_4$

$$w_5 = -\theta_1 w_4$$

$$w_5 = -\theta_1[\theta_1{}^3(1+\theta_1)]$$

$$w_5 = -\theta_1{}^4(1+\theta_1)$$

3.4 Invertibility model for IMA(1,2) model or First difference of MA(2) [to AR(∞) model]

$$y_t = y_{t-1} + \theta_1 e_{t-1} + \theta_2 e_{t-2} + e_t \tag{3.4}$$

$\nabla y_t = \theta_1 e_{t-1} + \theta_2 e_{t-2} + e_t$

$(1-B)y_t = (\theta_1 B + \theta_2 B^2 + 1)\ e_t$

Recall equation (2.24b)

$$e_t = \left(\sum_{j=0}^{\infty} w_j B^j\right) y_t$$

$$(1-B)y_t = (1+\theta_1 B + \theta_2 B^2)\left(\sum_{j=0}^{\infty} w_j B^j\right) y_t$$

$$(1-B) = (1+\theta_1 B + \theta_2 B^2)\ (1 + w_1 B^1 + w_2 B^2 + w_3 B^3 + w_4 B^4 + \cdots)$$

$$1 - B = 1 + w_1 B^1 + w_2 B^2 + w_3 B^3 + w_4 B^4 + \cdots + \theta_1 B + \theta_1 w_1 B^2 + \theta_1 w_2 B^3 + \theta_1 w_3 B^4 + \theta_1 w_4 B^5 + \cdots + \theta_2 B^2 + \theta_2 w_1 B^3 + \theta_2 w_2 B^4 + \theta_2 w_3 B^5 + \theta_2 w_4 B^6 + \cdots$$

Taking coefficient

B^1: $\quad -1 = w_1 + \theta_1$

$$w_1 = -(\theta_1 + 1)$$

B^2: $\quad 0 = w_2 + \theta_1 w_1 + \theta_2$

$$w_2 = -\theta_1 w_1 - \theta_2$$

$$w_2 = -(\theta_1 w_1 + \theta_2)$$

$$w_2 = \theta_1(\theta_1 + 1) - \theta_2$$

$$w_2 = \theta_1{}^2 + \theta_1 - \theta_2$$

B^3: $\quad 0 = w_3 + \theta_1 w_2 + \theta_2 w_1$

$$w_3 = -(\theta_1 w_2 + \theta_2 w_1)$$

$$w_3 = -[\theta_1[\theta_1(\theta_1 + 1) - \theta_2] + \theta_2(-(\theta_1 + 1))]$$

$$w_3 = -\left[\theta_1({\theta_1}^2 + \theta_1 - \theta_2) - \theta_1\theta_2 - \theta_2\right]$$

$$w_3 = -({\theta_1}^3 + {\theta_1}^2 - \theta_1\theta_2 - \theta_1\theta_2 - \theta_2)$$

$$w_3 = -{\theta_1}^3 - {\theta_1}^2 + 2\theta_1\theta_2 + \theta_2$$

B^4: $$0 = w_4 + \theta_1 w_3 + \theta_2 w_2$$

$$w_4 = -\theta_1 w_3 - \theta_2 w_2$$

$$w_4 = -\theta_1\left(-{\theta_1}^3 - {\theta_1}^2 + 2\theta_1\theta_2 + \theta_2)\right) - \theta_2\left({\theta_1}^2 + \theta_1 - \theta_2\right)$$

$$w_4 = {\theta_1}^4 + {\theta_1}^3 - 2{\theta_1}^2\theta_2 - \theta_1\theta_2 - {\theta_1}^2\theta_2 - \theta_1\theta_2 + {\theta_2}^2$$

$$w_4 = {\theta_1}^4 + {\theta_1}^3 - 3{\theta_1}^2\theta_2 - 2\theta_1\theta_2 + {\theta_2}^2$$

B^5: $$0 = w_5 + \theta_1 w_4 + \theta_2 w_3$$

$$w_5 = -\theta_1 w_4 - \theta_2 w_3$$

$$w_5 = -\theta_1\left({\theta_1}^4 + {\theta_1}^3 - 3{\theta_1}^2\theta_2 - 2\theta_1\theta_2 + {\theta_2}^2\right) - \theta_2(-{\theta_1}^3 - {\theta_1}^2 + 2\theta_1\theta_2 + \theta_2)$$

$$w_5 = -{\theta_1}^5 - {\theta_1}^4 + 3{\theta_1}^3\theta_2 + 2{\theta_1}^2\theta_2 - \theta_1{\theta_2}^2 + {\theta_1}^3\theta_2 + {\theta_1}^2\theta_2 - 2\theta_1{\theta_2}^2 - {\theta_2}^2$$

$$w_5 = -{\theta_1}^5 - {\theta_1}^4 + 4{\theta_1}^3\theta_2 + 3{\theta_1}^2\theta_2 - 3\theta_1{\theta_2}^2 - {\theta_2}^2$$

Table 1: Summary Table of the Generalization Invertibility Models: MA(1), MA(2), IMA(1,1) and IMA (1,2) to AR(∞) models

AR(∞) $parameters$	MA(1)	MA(2)	IMA(1,1)	IMA (1,2)
w_1	$-\theta_1$	$-\theta_1$	$-(1+\theta_1)$	$-(1+\theta_1)$
w_2	$\theta_1{}^2$	$\theta_1{}^2-\theta_2$	$\theta_1\,(1+\theta_1)$	$\theta_1{}^2+\theta_1-\theta_2$
w_3	$-\theta_1{}^3$	$-\theta_1{}^3+2\theta_1\theta_2$	$-\theta_1{}^2\,(1+\theta_1)$	$-\theta_1{}^3-\theta_1{}^2+2\theta_1\theta_2+\theta_2$
w_4	$\theta_1{}^4$	$\theta_1{}^4-3\theta_1{}^2\theta_2+\theta_2{}^2$	$\theta_1{}^3\,(1+\theta_1)$	$\theta_1{}^4+\theta_1{}^3-3\theta_1{}^2\theta_2-2\theta_1\theta_2+\theta_2{}^2$
w_5	$-\theta_1{}^5$	$-\theta_1{}^5+4\theta_1{}^3\theta_2-3\theta_1\theta_2{}^2$	$-\theta_1{}^4\,(1+\theta_1)$	$-\theta_1{}^5-\theta_1{}^4+4\theta_1{}^3\theta_2+3\theta_1{}^2\theta_2-3\theta_1\theta_2{}^2-\theta_2{}^2$
	$(-\theta_1)^r$	Apply Pascal triangle	$(-1)^r(\theta_1)^{r-1}(1+\theta_1)$	Apply Pascal triangle

Generalization of the obtained invertibility models

3.5 From MA(1) to AR(∞) models

$$y_t = \sum_{j=1}^{\infty} w_j y_{t-j} + e_t \qquad (3.5a)$$

by substitution $w_j \ as(-\theta_1)^r$

$$y_t = \sum_{r=1}^{\infty} (-\theta_1)^r y_{t-r} + e_t \qquad (3.5b)$$

3.6 From MA(2), to AR(∞) models

Apply Pascal triangle ideal to obtain $w_i, i = 1, 2, \ldots$:

Add both values and multiply by (-1)

-1				
1	-1			
-1	2			
1	-3	1		
-1	4	-3		
1	-5	6	-1	
-1	6	-10	4	
1	-7	15	-10	1
-1	8	-21	20	-5

etc.

3.7 From IMA(1,1) to AR(∞) models

$$y_t = \sum_{r=1}^{\infty} (-1)^r (\theta_1)^{r-1} (1+\theta_1) y_{t-r} + e_t \qquad (3.6)$$

3.8 From IMA(1,2) to AR(∞) models

Apply Pascal triangle ideal to obtain $w_i, i = 1, 2, \ldots$:

-1	-1					
1	1	-1				
-1	-1	2	-1			
1	1	-3	-2	1		
-1	-1	4	3	-3	-1	
1	1	-5	-4	-1	6	3

etc.

4. DISCUSSION OF R*E*SULTS

4.1 Establish the invertibility model from MA(1), MA(2), IMA(1,1), IMA(1,2) to AR(∞) model

The study obtained the invertibility model for MA(1), MA(2), IMA(1,1), IMA(1,2), using the backward shift notation and repeated substitution method. Here, some constraints were imposed on the MA parameters to convert to an AR(∞) process, the most recent error was written as a linear function of the current and past observations. Coefficients were equated to obtain the parameter estimates; these obtained parameter estimates were now substituted into the invertibility stationary model for AR(∞) process to obtain the inverted models for MA(1) and MA(2). For IMA(1) and IMA(2), the differencing operator was introduced to make the series stationary before imposing the constraints, obtaining the parameter estimates and substituting into the invertibility stationary model for AR(∞) process to obtain the inverted models for IMA(1) and IMA(2). See sections 3.1.1, 3.1.2, 3.1.3 and 3.1.4. Table 1 showed the summary of MA(1), MA(2), IMA(1,1) and IMA (1,2) models to AR(∞) models. This is in tandem with Granger and Allan (1978), who proposed a generalized definition of invertibility and applied it to linear, non-linear and bilinear models.

4.2 Generalization of the established invertibility models' parameters from the selected MA(1), MA(2), IMA(1,1), IMA(1,2) models to AR(∞) models

Here, the study obtained the order of q needed for the inverted model by substituting the w_j (i.e the parameter estimates) as $(-\theta_1)^r$ for MA(1) and IMA(1,1) from the summary table 4.1. The MA(2) and IMA(1,2) follow the idea of Pascal triangle see sections 3.1.6 and 3.1.8

5 CONCLUSION

This study inverted the univariate moving average time series model for forecasting accuracy. The inverted models were obtained from Moving Average 'MA' (invertibility model) for the first two lags MA(q) process i.e. MA(1), MA(2), IMA(1,1), IMA(1,2) and the parameters of the AR(∞) models have been generalized. The implication of this result was that converting from MA(q) model to infinite AR model depend on the MA(q) model parameters. The MA parameters must be subjected to the invertibility condition.

REFERENCES

Adhikari, R. & Agrawal, R. K. (2013). An introductory study on time series modeling and forecasting. *arXiv preprint arXiv:1302.6613*.

Box, G. & Jenkins G. (1970). *Time Series Analysis: Forecasting and Control*. San Francisco: Holden-Day. 156-207.

Box, G., Cox, D. & Draper, N. (2015). *Analysis of Variance, Design, and Regression: Linear Models*. Wiley. 234-305.

Box, G., Jenkins, G. & Reinsel, G. (2008). *Time Series Analysis: Forecasting and Control*. Wiley. 189-245.

Box, G., Jenkins, G., Reinsel, G. & Ljung, G. (2015). *Time Series Analysis: Forecasting and Control*. John Wiley & Sons. 190-257.

Brockwell, P. & Davies, R. (2002). *Introduction to Time Series and Forecasting*. Second Edition. Springer-Verlag, New York. 359-402.

Brockwell, P. & Davis, R. (2016). *Time Series: Theory and Methods.* Springer. 521-589.

Chan, N. H., & Palma, W. (2015). *Threshold autoregressive models*. Handbook of Financial Time Series, 113-132.

Granger, C. & Allan A. (1978). On the invertibility of time series models. *Stochastic Processes and their Applications*, *8*(1), 87-92. https://doi.org/10.1016/0304-4149(78)90069-8.

Hamilton, J. D. (1994). *Time Series Analysis*. Princeton University Press. 167-192

Hyndman, R. & Athanasopoulos, G. (2018). *Forecasting: Principles and Practice*. OTexts. 191-256

Hyndman, R. J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice* (3rd ed.). OTexts. 234-287

MA models. *Journal of Forecasting*. https://onlinelibrary.wiley.com/

Lütkepohl, H. (2005). *New Introduction to Multiple Time Series Analysis*. Springer. 278-395

Puthran, D., Shivaprasad, H. C., Kumar, K. K., & Manjunath, M. (2014). Comparing SARIMA and Holt-Winters' forecasting accuracy with respect to Indian motorcycle industry. *Transactions on Engineering and Sciences*, *2*(5), 25-28.

Shumway, R. & Stoffer, D. (2017). *Time Series Analysis and Its Applications: With R Examples*. Springer. 194-234

Tibshirani, R. (1996). Regression Shrinkage and Selection via the Lasso. *Journal of the Royal Statistical Society*, Series B (Methodological), *58*(1), 267-288.