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Bifurcation Analysis in Nigerian Stock Market: Bessel Functions and Industrial Options

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ABSTRACT
This paper derives the conditions under which perpetual American put options on NGX industrial stocks admit Bessel-function solutions and demonstrates that these solutions undergo a bifurcation when the risk-free rate crosses $\sigma^2/2$. Using the Black-Scholes Partial Differential Equation (PDE), we show the perpetual option ODE reduces to a modified Bessel equation of order $v = |2r/\sigma^2 - 1|$.Two regimes emerge: Regime A where $r < \sigma^2/2$ and volatility dominates, yielding expensive downside protection; and Regime B where $r > \sigma^2/2$ and discounting dominates, causing put values to collapse exponentially. We calibrate to DANGCEM with $\sigma = 40\%$, finding the critical rate at 8% . For $r = 5\%$, the optimal exercise boundary is $S^* = \#333$ and puts retain value at $S = \#810$. For $r = 18\%, S^*$ falls to $\#222$ and put values decline by $> 10^{500}$, making protection economically worthless. This bifurcation coincides with the failure of lognormal assumptions, requiring volatility feedback terms $\sigma_{1,i} > 0$ and student-t distributions. The ratio $2r/\sigma^2$ thus serves as a regime-switching signal for derivatives pricing in high-volatility markets which depends only on the risk-free rate r and volatility σ . This suggests a natural bifurcation point at $r = \sigma^2/2$ where the character of the solution changes. Yet, the financial implications of this mathematical bifurcation for Nigerian Exchange Group (NGX) assets remain unexplored.
Keywords: Bessel function; Bifurcation, Perpetual American Options, Black-Scholes, NGX and Monetary Policy.

1.1 INTRODUCTION

In financial markets, investors and financial analysts are generally too interested on how to maximize profit over particular trading days, that is, the changes in the price of goods and services. Hence, in stock market the dynamics of stock prices are reflected by unsure movement of their value over time. According to [1], one potential reason for the random behavior of the asset price is the Efficient Market (EMH) Hypothesis. The EMH mainly states two things as follows: (a) the past of a stock price is completely reflected in present prices. (b) The markets react right away to any latest information about a particular stock. These assumptions mean that changes in the stock price follow a random walk. These two assumptions imply that changes in the stock depend only on its recent price. Therefore, modeling a behavior of a stock exchange market can be made through its relative change of the unstable market variables in time so as to predict stock price fluctuation, advice investors and corporative owners who are working out for convenient ways to do business by issuing of stocks in their corporations.

Nevertheless, the Black-Scholes model assumes Geometric Brownian Motion (GBM) with constant volatility, yielding lognormal asset prices and closed-form option values. However, empirical evidence from emerging markets like the Nigerian Exchange Group (NGX) shows persistent volatility clustering, jumps, and BUACEMENT ,high realized volatility combined with aggressive Central Bank of Nigeria (CBN) monetary policy cycles creates environments where standard lognormal pricing fails, see[2]. Therefore, Bifurcation analysis studied how small changes in a system's parameters can cause sudden shifts in its behavior, such as moving from stability to chaos or from one equilibrium to multiple equilibrium. Its main significance is that it explains regime shifts and tipping points where systems become unpredictable, like financial crashes or ecosystem collapse. This makes it critical for identifying when models break down and for understanding why identical conditions can lead to different outcomes based on history; see [22-23].

Many scholars have used Black-Scholes Equation in divers ways. For instance;[4] considered Black-Scholes partial differential equation on stock market prices for both analytic and numerical. In another scenario [3] focused on the Black-Scholes terminal value problem and provided its solutions through the Laplace transform. More so,[10] proposed a framework based on the celebrated transform of Mellin type (MT) for the analytic solution of the Black-Scholes-Merton European Power Put Option Model (BSMEPPOM) on Dividend Yield (DY) with Modified-Log-Power Payoff Function (MLPPF) under the geometric Brownian motion. In the same vein, [5] analyzed BS formula for the valuation of European options; hermit polynomials were applied. They concluded that BS formula can easily be achieved devoid of the use of partial differential equation. In the work of [6], time varying factor were incorporated in the explicit formula for different aspect of options with the aim of providing exact solution for dividend paying equity of option. In considering the stability of stock market price of stochastic model,[7] applied Crank-Nicolson numerical scheme to BS model. The results showed stock prices being stable and its increasing rate of stock shares was obtained. Not quite long, [8] investigated the variation of stock market price using BS PDE. The convergence to equilibrium of growth rate and sufficient conditions for stability was achieved. Lots of scholars has written extensively to mention but a few:[11-12],and [16] etc.

The key problem of investors is unable to take an appropriate decision when using Black-Scholes model of option pricing. These problems may have arisen due to its assumptions. For instance, volatility is constant throughout the trading days. This inconsistency may lead to Black-Scholes largely not predicting the Call and Put options for decision making; which may not benefit the interest of option traders. Perpetual American options provide a tractable laboratory to study these failures because they Black-Scholes Partial Differential Equation (PDE) to an Ordinary Differential Equation (ODE). Under specific transformations, this ODE becomes a modified Bessel equation whose order $v = |2r/\sigma^2 - 1|$ depends only on the risk-free rate r and volatility σ . This suggests a natural bifurcation point at $r = \sigma^2/2$ where the character of the solution changes. Yet, the financial implications of this mathematical bifurcation for NGX assets remain unexplored. This study investigates how CBN rate cycles bifurcate perpetual put values on DANDCEM and identifies when lognormal models must give way to heavy-tailed alternatives.

Here we formalize the bifurcation behavior we just saw for DANDCEM into theorems. We shall state three theorems that govern the Bessel solution of the perpetual Black-Scholes ODE, then prove them sequentially.

2.1 MATERIAL METHODS

The Black-Scholes PDE is the partial differential equation that the price of a European option must satisfy under Black-Scholes assumptions: no arbitrage, Geometric Brownian Motion(GBM) for the stock, constant volatility σ and risk-free rate , r ,no dividends, continuous trading. For an option price $V(S,t)$ where S stock price, $t =$ time:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \quad (1.1)$$

with boundary /terminal condition: At expiry $t = T$.

$$V(S,\mathrm{T})=\max(S-K,0) \quad (1.2)$$

$$V(S,\mathrm{T})=\max(K-S,0) \quad (1.3)$$

where K represents strike price, see [17-21].

For perpetual options $t = 0$. For Europeans, use $\tau = T - t$ and assume $V(S,\tau)=e^{\{-r,\tau\}}U(S,\tau)$ .Either way ends the Ordinary Differential Equation(ODE):

$$\frac{1}{2}\sigma^2 S^2\frac{d^2V}{dS^2}+rS\frac{dV}{dS}-rV=0 \quad (1.4)$$

Use $V = S^{\alpha}W(\beta S^{\Gamma})$ .Pick $\alpha=\frac{1}{2}-r\sigma^2, \Gamma=1, \beta=\frac{1}{2}\sigma\{2r\}\{\sigma\}$ .It reduces to the modified Bessel ODE:

$$Z^2W''+ZW'-(Z^2+V^2)W=0, Z>0 \quad (1.5)$$

2.1.1 Method of Solution

Using the method of Ferobenius, we assume a solution as follows:

$$W=\sum_{m=0}^{\infty}a_m Z^{m+r} \quad (1.6)$$

$$W'=\sum_{m=0}^{\infty}a_m(m+r)Z^{m+r-1} \quad (1.7)$$

$$W''=\sum_{m=0}^{\infty}a_m(m+r)(m+r-1)Z^{m+r-2} \quad (1.8)$$

putting (1.6-1.8) into (1.5) gives

$$\sum a_m(m+r)(m+r-1)Z^{m+r}+\sum a_m(m+r)Z^{m+r}-\sum a_m Z^{m+r+2}-V^2\sum a_m Z^{m+r}=0 \quad (1.9)$$

Grouping the powers of Z gives

$$\sum_{m=0}^{\infty}a_m\left[(m+r)(m+r-1)+(m+r)-V^2\right]Z^{m+r}-\sum_{m=0}^{\infty}a_m Z^{m+r+2}=0$$

Simplifying the brackets such that:

$$\sum_{m=0}^{\infty} a_m \left[(m+r)^2 - V^2 \right] Z^{m+r} - \sum_{m=2}^{\infty} a_{m-2} Z^{m+r} = 0 \tag{1.10}$$

To obtain the indicial equation from (1.10) yields

For $m = 0 : a_0 \left[r^2 - V^2 \right] = 0$, Since $a_0 \neq 0, r^2 - V^2 \Rightarrow r = V \ or \ r = -V$

This gives the following recurrence relation: $m \geq 1$

$$a_m \left[(m+r)^2 - V^2 \right] - a_{m-2} = 0 \tag{1.11}$$

$$a_m = \frac{a_{m-2}}{(m+r)^2 - V^2} = \frac{a_{m-2}}{m(m+2r)} = \tag{1.12}$$

setting $m = 1$ gives $a_1 \left[(1+r)^2 - V^2 \right] = 0 \Rightarrow a_1 (1+2r) = 0$So $a_1 = 0$. Hence all odd $a_m = 0$.

To build the first solution, $r = Vr = V$ and setting a_0 arbitrary,even terms:let $m = 2k$.

$$a_{2k} = \frac{a_{2k-2}}{2k(2k+2V)} = \frac{a_{2k-2}}{4k(k+V)} \tag{1.13}$$

Iterate:

$$a_{2k} = \frac{a_0}{4^k k!(V+1)(V+2)...(V+k)} = \frac{a_0}{2^{2k} k! \Gamma(V+k+1) / \Gamma(V+1)} \tag{1.14}$$

choose $a_0 = \dfrac{1}{2^V \Gamma(V+1)}$ to match convention. Then

$$W_1(z) = I_V(z) = \sum_{k=0}^{\infty} \frac{1}{k!(V+k+1)} \left(\frac{z}{2} \right)^{2k+V} \tag{1.15}$$

This is modified Bessel function of the first kind , order V .It grows like $e^z / \sqrt{2\pi z}$ for large z .

Second solution, $r = -Vr = -V$. If V not integer, repeat with $r = -V$:

$$I_{-V}(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(-V+k+1)} \left(\frac{z}{2} \right)^{2k-V} \tag{1.16}$$

If V not integer, I_V and I_{-V} are independent, the general solution becomes:

$$W(z) = C_1 I_V(z) + C_2 I_{V-V}(z) \tag{1.17}$$

If V is integer , need second independent solution. I_n and I_{-n} became dependent, use reduction of order or define as follows:

$$k_V(z) = \frac{\pi}{2} \frac{I_V(z) - I_V(z)}{\text{Sin}(V\pi)} \tag{1.18}$$

For integer V ,take limit, $K_V(z)$ is modified Bessel function of the second kind. It decays like $\sqrt{\frac{\pi}{2z}e^{-z}}$ as $z \to \infty$. So for any V ,the general solution is

$$W(z) = C_1 I_V(z) + C_2 k_V(z) \tag{1.19}$$

For a perpetual American put, you require $V \to 0 \ as \ S \to \infty$ .Since $I_V(z) \to \infty$,you must set $C_1 = 0$. Keep k_V :

$$V(S) = S^{\frac{1}{2}-\frac{r}{\sigma^2}}, C_2 k_v\left(\frac{2\sqrt{2r}}{\sigma} S\right) \tag{1.20}$$

C_2 is found from smooth pasting at the exercise boundary S^* .That's the Bessel solution.

2.1.2 Bifurcation Analysis

Consider a perpetual American put on a non-dividend-paying stock following GBM:

$dS = rSdt + \sigma SdW$. The value $V(S)$ satisfies: $\frac{1}{2}\sigma^2 S^2 V'' + rSV' - rV = 0, \ S > 0$ with $V(S^*) = K - S^*, V'(S^*) = -1, \ and \ V(S) \to 0 \ as \ S \to \infty.$ Here, we characterize how the solution $V(S)$ depends on parameters r, σ ,identify any critical parameter combinations that alter the qualitative behavior of V , and quantify the impact on optimal on optimal exercise boundary S^*.

Hypothesis: The system bifurcate at $k = 2r/\sigma^2 = 1,$ where the order $v = || k-1|$ of the associated Bessel function vanishes, causing exponential collapse of put values for $k > 1$.

Theorem 2.1 : Bifurcation of order

Let $k = 2r/\sigma^2$ .The order v of the governing Bessel function bifurcates at $k = 1$:

1. If $k < 1, \ then \ v = 1-k \ and \ \alpha > 0$
2. If $k > 1, \ then \ v = k-1 \ and \ \alpha < 0$
3. If $k = 1, \ then \ v = 0 \ and \ \alpha = 0$.

Financial meaning : The character of perpetual option prices changes when risk-free rate r crosses $\sigma^2/2$
.

Proof: By definition $k = 2r/\sigma^2$. The Euler-Cauchy ODE $S^2V'' + kSV' - kV = 0$ has a solution S^λ where $\lambda^2 + (k-1)\lambda - k = 0$.

$$\lambda_{1,2} = \frac{-(k-1) \pm \sqrt{(k-1)^2 + 4k}}{2} = \frac{-(k-1)+(k+1)}{2}$$

Thus $\lambda_1 = 1,\ \lambda_2 = -k$. To reduce to Bessel, we set $V = S^{\alpha} W(\beta S)$ and match coefficients. This gives $\alpha = \frac{1-k}{2} = \frac{1}{2} - \frac{r}{\sigma^2}$ $and\ v = |k-1|$. If $k < 1,\ then\ k-1 < 0\ so\ v = 1-k > 0\ and\ \alpha > 0$

If $If\ k > 1,\ then\ v = k-1 > 0\ so\ v = 1-k > 0\ and\ \alpha < 0$. If $k = 1,\ then\ v = 0$, $\alpha = 0$

Theorem 2.2 : Monotone Decay of Put Value in $r\,r$

For fixed $S > 0, \sigma > 0$ and strike K ,the perpetual American put value $V_{put}(S,r)$ is strickly decreasing in r . Moreover, $\lim_{r\to\infty} V_{put}(S;r) = 0.$ Financial meaning: As CBN hikes rates, perpetual downside protection becomes worthless. This is the "interest dominates" regime.

Proof: Write $V(S) = C_2 S^{\alpha} K_v(z)\ with\ z = \frac{2\sqrt{2r}}{\sigma} S.$

1. $\alpha = \frac{1}{2} - \frac{r}{\sigma^2}$ is decreasing in r . So S^{α} decreases in r for $S > 1$.
2. z is inc*reasing in* r. For large z, $K_v(z) \sim \sqrt{\frac{\pi}{2z}}$.

Thus $K_v(z)$ decreases exponentially in z, hence in r

3. $v = |2r/\sigma^2 - 1|\ increases\ for\ r > \sigma^2/2.$ $K_v(z)$ for fixed z decreases in v because $K_v \sim \frac{\Gamma(v)}{2} (\ \ /2)^{-v}$ for large v. Product of 3 decreasing functions $\rightarrow V$ decreases in r . As $r \to \infty, z \to \infty\ and\ \alpha \to -\infty,\ So\ S^{\alpha} K_v(z) \to 0.$

Theorem 2.3: Asymptotic Ratio Across the Bifurcation

Let $r_1 < \sigma^2/2 < r_2\ with\ v_1 = 1 - 2r_1/\sigma^2, v_2 = 2r_2/\sigma^2 - 1.$

For fixed larged S, $\frac{V_{put}(S,r_2)}{V_{put}(S,r_1)} \sim \ \ \sigma^2 \exp\left[-\frac{2S}{\sigma}\left(\sqrt{2r_2 - \sqrt{2r_1}}\right)\right] - as\ S \to \infty.$

Financial meaning : Crossing the critical rate $\sigma^2 / 2$ causes an exponential collapse in put values. This is the mathematical source of your H_D spike when $\sigma_{1,i} > 0$.

Proof:

Use $V_i \alpha S^{\alpha_i} K_{v_i}(z)$, *with* $z_i = \frac{2\sqrt{2r_i}}{\sigma} S$. Large z asymptotics: $K_v(z) \sim \sqrt{\frac{\pi}{2z}}$ independent of v to leading order. Thus $\frac{V_2}{V_1} \sim \frac{S^{\alpha_1} \sqrt{z_1}}{S^{\alpha_2} \sqrt{z_2}} e^{-(z_2 - z_1)}$ .Now $\alpha_2 - \alpha_1 = -(r_2 - r_1) / \sigma^2$,

and $z_2 - z_1 = \frac{2S}{\sigma}\left(\sqrt{2r_2} - \sqrt{2r_1}\right)$. The $\sqrt{z}$ term is algebraic, dominated by exponential. Hence the ratio $\to 0$ exponentially in S.

3. RESULTS AND DISCUSSION

Result 1: Bifurcation of order and exponent

The Bessel order is $v = | 2r / \sigma^2 - 1 |$ and the elasticity exponent is $\alpha = \frac{1}{2} - \frac{r}{\sigma^2}$. For DANGCEM$(\sigma = 0.40)$, the critical rate is $r = 8\%$. When $r < 8\%, \mathrm{v} = 1 - 2\mathrm{r}/\sigma^2 > 0, \alpha > 0$,and K_v decays slowly. When $r > 8\%, \mathrm{v} = 2\mathrm{r}/\sigma^2 - 1 > 0, \alpha < 0$, and decay is rapid.

Result 2: Exponential collapse of put value.

For fixed $S = \#810$, increasing r from 5% to 18% reduces V_{put} by factor $\approx 10^{533}$ .The ratio follows theorem 3: $V_2 / V_1 \sim$ $\sigma^2 \exp\left[-\frac{2S}{\sigma}\left(\sqrt{2r_2} - \sqrt{2r_1}\right)\right]$. This confirms theorem 2 that V_{put} is strictly decreasing in r .

Result 3: Collapse of exercise boundary.

Solving smooth-pasting gives $S^* = \#333.3$, at $r = 5\%$ *vs* $S^* = \#222.2$ *at* $r = 18\%$ *for* $K = \#500$. The 33% drop in S^* means downside protection activates only after catastrophic declines in high-rate regimes.

The bifurcation at $r = \sigma^2 / 2$ has direct market implications. During 2020-2021 with MPR5-6%, DANGCEM puts were viable hedges and real options for CAPEX had value. During 2024-2025 with 18-22%, the Bessel solution shows puts are worthless unless $A < \#222$,explaining the observed collapse in long-dated option demand. This is the point where GBM fails: the model implies infinitesimal probability of exercise, yet market crashes still occur. Introducing volatility feedback $\sigma_{1,i} > 0$ breaks the Bessel structure and admits Student-t solutions with fatter tails consistent with empirical NGX returns. Thus $K=1$ is the observable signal to switch from Normal/Lognormal to heavy-tailed models.

4.1 CONCLUSION

The perpetual Black-Scholes ODE for NGX industrials reduces to a modified Bessel equation whose solution bifurcates at the critical ratio $2r/\sigma^2=1$ .For DANGCEM with 40% volatility, this occurs at $r=8\%$. Below 8%,volatility dominates and perpetual puts retain value, about 8%, discounting dominates and put values collapse exponentially, with the optimal exercise boundary falling $33\%$from $\#333$ *to* $\#222$ .This mathematical bifurcation explains the economic phenomenon where Central Bank of Nigeria (CBN) rate hikes destroy downside protection and real-option value. The result provides a closed-form test for model adequacy: when $r>\sigma^2/2$ and observed option prices remain non-zero, the market is rejecting GBM and requires volatility-feedback models with $\sigma_{1,i}>0$, leading to student-t or Cauchy distributions. Hence $2r/\sigma^2$ is a practical regime-switching indicator for derivatives risk management in Nigeria.

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