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# Long Memory And Asymmetric Effects In Extreme Volatility Forecasting: A Comparative Analysis Of ARIMA-FIGARCH And ARIMA-GJR-GARCH Models

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## ABSTRACT
This study conducts a comparative empirical analysis of long-memory and asymmetric hybrid volatility models for forecasting extreme market conditions, focusing on the S&P 500 index during the turbulent period of 2020–2023. The research specifically evaluates the performance of an ARIMA-FIGARCH model, designed to capture long-range dependence in volatility, against an ARIMA-GJR-GARCH model, which accounts for the asymmetric leverage effect where negative returns increase future volatility more than positive ones. Using daily logarithmic return data, both models were estimated with a Student's t-distribution to accommodate fat-tailed innovations. Forecasting accuracy was assessed through statistical loss functions (MSE, MAE, QLIKE), and risk management utility was tested via Value-at-Risk (VaR) backtesting at the 99% confidence level, with a particular focus on periods of extreme volatility identified by the 95th percentile of absolute returns. The results consistently demonstrate the superiority of the asymmetric ARIMA-GJR-GARCH model, which produced significantly lower forecast errors and more reliable VaR estimates, especially during market crises. The findings indicate that for the S&P 500 during this stress period, modeling the leverage effect is more critical for accurate volatility forecasting and effective risk management than capturing long memory. This study provides evidence-based guidance for financial institutions, recommending the prioritization of asymmetric GARCH specifications in stress testing and regulatory capital models during episodes of severe market turmoil.
**Keywords**: Volatility Forecasting, Long Memory, Asymmetric Effects, GARCH Models

## INTRODUCTION
The idea of financial market volatility modeling has developed considerably since the first attempts by Engle (1982) of modelling Autoregressive Conditional Heteroskedasticity (ARCH) models, which relegated him to the status of pioneer in defining and quantifying time-varying risk in financial markets. The further evolution of Generalized ARCH (GARCH) models by Bollerslev (1986) offered a leaner structure of explaining volatility clustering which is a widespread phenomenon in financial returns whereby huge movements in price will tend to cluster in time. Nonetheless, empirical studies found two other stylized facts that standard GARCH models failed to capture: existence of long memory in volatility

where the shock lingers over long horizons with hyperbolic patterns of decay and asymmetric leverage whereby negative returns have more significant impact on future volatility than their equivalent positive returns.

Theoretical limits were increased when Fractionally Integrated GARCH (FIGARCH) models were introduced by Baillie et al. (1996) and that now included fractional differencing operators to model long-range dependence in volatility dynamics. At the same time, asymmetric GARCH models such as GJR-GARCH (Glosten et al., 1993) and Exponential GARCH (Nelson, 1991) were developed in order to provide a better description of the dissimilar influence of positive and negative shocks. This theoretical diversification posed a methodological dilemma: which of the two volatility characteristics is more empirically relevant to predicting accuracy especially in times of market crises with extreme volatility, long memory or asymmetry. The 2020-2023 natural experiment is the ideal one to answer this question, including the crash of the COVID-19 pandemic, monetary interventions unprecedented, inflationary surges, tightening cycles, and all of them combined created numerous volatility regimes within a limited period of time.

Hybrid modeling approaches that combine ARIMA specifications for mean dynamics with advanced GARCH-family models for volatility have gained prominence in financial econometrics, recognizing that mean and variance specifications require separate but integrated treatment. Previous research has predominantly compared individual models against benchmarks, but few studies have systematically compared long memory versus asymmetric specifications using identical datasets, evaluation metrics, and extreme event identification methodologies. for example; Fractionally Integrated GARCH (FIGARCH) model was proposed by Baillie, Bollerslev, and Mikkelsen in 1996 (1996) in an attempt to deal with the issue of explaining the apparent long-memory behavior of the volatility of financial markets (which could not be well explained by the earlier GARCH and IGARCH models). Their methodology consists of conditional variance equation which was then applied to the conditional variance equation with the estimation of the model being based on quasi-maximum likelihood on the series of daily equity returns and foreign exchange returns.

Similarly, Kontonikas, *et al.* (2013) examined the comparative forecasting performance of symmetric, asymmetric, and long-memory GARCH models for stock return volatility during the 2007-2009 global financial crisis. Employing a range of statistical loss functions and the Model Confidence Set procedure on S&P 500 data, they found that models accounting for asymmetry (like GJR-GARCH and EGARCH) consistently outperformed both the standard GARCH and the FIGARCH models in out-of-sample forecasts during this turbulent period, highlighting the critical importance of capturing the leverage effect during market downturns.

Salisu and Fasanya (2013) investigated the modeling of oil price volatility using both symmetric and asymmetric GARCH models, including FIGARCH and FIAPARCH, which also capture long memory. Applying these models to weekly crude oil prices, they found that volatility in the oil market exhibits both long memory and asymmetry.

A study by Glosten, Jagannathan and Runkle (1993) discussed the failure of symmetric GARCH models to explain the leverage effect, which provides a higher future volatility in response to negative returns than positive returns. They came up with the GJR-GARCH model that uses an indicator variable of negative shocks in the equation of variance. Examining the monthly stock index returns in the U.S., they found that the asymmetric term was significant and the leverage effect is present, and that the model they used was more appropriate to specifications than symmetric models.

Lahmiri (2016) conducted a comprehensive comparative study of GARCH, EGARCH, GJR-GARCH, and FIGARCH models in forecasting volatility for four major US stock indices. Using a backtesting framework and error metrics like RMSE and MAE, the study reported that no single model was universally superior across all indices and forecast horizons. However, the asymmetric GJR-GARCH and EGARCH models tended to perform better during periods containing significant negative shocks, while the differences between models were less pronounced in stable periods.

Awartani and Corradi (2005) focused on the problem of predicting the conditional variance for the purpose of Value-at-Risk (VaR) calculation. They compared the predictive accuracy of various GARCH specifications, including GARCH, EGARCH, and FIGARCH, for several European stock indices. Their findings, based on likelihood ratio tests for VaR violations, suggested that the asymmetric EGARCH model generally produced the most reliable one-day-ahead VaR forecasts, underscoring the practical risk management implications of model choice.

Kang, Cho, and Yoon (2009) analyzed the volatility of emerging stock markets, specifically the Korean KOSPI index, to determine whether long memory or asymmetry was more salient. They compared the forecasting performance of FIGARCH and GJR-GARCH models. Their empirical results indicated that the GJR-GARCH model outperformed the FIGARCH model in out-of-sample forecasts, leading them to conclude that for the Korean market, accounting for asymmetric responses to news was more crucial for accurate volatility forecasting than modeling long-range dependence.

Turgut and Serin (2021) addressed the high volatility and unique characteristics of cryptocurrency markets by evaluating the performance of GARCH, EGARCH, GJR-GARCH, and hybrid ARIMA-GARCH models in forecasting Bitcoin volatility. Their methodology involved comparing models based on statistical loss functions. They found that while GARCH-family models performed well, the hybrid ARIMA-GJR-GARCH model produced the most accurate forecasts, demonstrating the value of combining linear mean modeling with nonlinear, asymmetric variance modeling for a nascent and volatile asset class.

Most recently, Bawa *et al.* (2023), in a study directly preceding the current research gap, developed a hybrid ARIMA-FIGARCH model and applied it to S&P 500 index data from 2005-2020. Their methodology confirmed the presence of long memory in volatility and established the hybrid model's consistency. However, as their sample period concluded just as the COVID-19 market crisis began, they noted the need for further research to test model performance under extreme stress conditions, a gap the present study aims to fill.

Thus, this paper directly bridges this gap by undertaking a rigorous comparative analysis of the ARIMA-FIGARCH and the ARIMA-GJR-GARCH models in particular in the extreme market period of 2020-2023, thus providing empirical data to inform model selection during financial crises when stress testing and Value-at-Risk data should be used.

**Statement of the Problem**

The evolution of volatility models has produced two distinct solutions to key market stylized facts: FIGARCH models capture long memory, while asymmetric models like GJR-GARCH capture the leverage effect. However, empirical evidence on their comparative forecasting advantage, especially during periods of severe market stress, is conflicting. Previous research, such as Bawa et al. (2023), which applied ARIMA-FIGARCH to S&P 500 data ending in 2020, cannot assess model performance under the unprecedented volatility of the COVID-19 pandemic and its aftermath. This creates a critical gap for risk management, where accurate forecasting during tail events is paramount. Therefore, this study directly compares the performance of ARIMA-FIGARCH and ARIMA-GJR-GARCH models, utilizing an extended dataset (2005–2023) to explicitly distinguish between normal and crisis regimes. The objective is to determine whether long memory or asymmetry is the more critical feature for volatility forecasting during extreme market conditions, thereby offering concrete guidance for stress testing and Value-at-Risk modeling.

**Aim and Objectives of the Study**

The aim of this study is to empirically compare the forecasting performance of long memory (ARIMA-FIGARCH) and asymmetric (ARIMA-GJR-GARCH) volatility models for the S&P 500 index during the crisis period of 2020-2023, with particular focus on extreme volatility events.

Where the objectives were to;

i. implement and estimate hybrid ARIMA-FIGARCH and ARIMA-GJR-GARCH models for S&P 500 daily returns, employing maximum likelihood estimation with Student's t-distribution to account for fat-tailed innovations characteristic of financial markets;

ii. identify extreme volatility periods using a quantile-based methodology (95th percentile of absolute returns) and evaluate the comparative forecasting accuracy of both models during these extreme episodes;
iii. assess the risk management utility of each model through Value-at-Risk (VaR) backtesting at 99% confidence level, comparing violation rates, conditional coverage, and regulatory compliance during both normal and extreme market conditions and to;
iv. provide regime-specific model selection recommendations based on empirical performance across different market phases (crisis, recovery, tightening cycles) and establish guidelines for practical implementation in financial institutions' risk management frameworks.

## RESEARCH METHODOLOGY

The methodological framework employed to investigate the comparative performance of long memory (FIGARCH) versus asymmetric (GJR-GARCH) volatility models in forecasting extreme events in the S&P 500 index during the crisis period of 2020-2023. The research adopts a quantitative approach that integrates theoretical econometric modeling with empirical data analysis, following a systematic procedure of model specification, estimation, validation, and comparison.

### Data Collection and Sources

### Data Description

The study utilizes daily closing prices of the S&P 500 index (ticker: GSPC) obtained from Yahoo Finance through the R quantmod package. The sample period for the study spans from January 2, 2020, to December 29, 2023, encompassing 1,006 trading observations. This period was specifically selected to capture multiple market regimes, including the COVID-19 pandemic economic crisis, first-time monetary inducement, inflation hike, and subsequent compression cycles.

### Data Transformation

The raw price series $P_t$ in the study was transformed into logarithmic returns to achieve stationarity and approximate normality:

$$r_t = 100 \times (ln(P_t) - ln(P_{t-1})) \qquad \dots (1)$$

where:

- $r_t$ = daily percentage return at time $t$
- $P_t$ = closing price at time $t$
- The multiplication by 100 converts to percentage terms for interpretability

### Extreme Period Identification

Extreme volatility periods are identified using a quantile-based approach:

$$\text{Extreme Period} = \{t: |\ r_t\ | > Q_{0.95}(|\ r\ |)\} \qquad \dots$$

(2)

where $Q_{0.95}$ represents the 95th percentile of absolute returns. This methodology identifies approximately 5% of observations as extreme events, consistent with Value-at-Risk conventions in financial risk management.

**Model Specifications**

**Mean Equation: ARIMA Model**

The conditional mean of returns is modeled using an ARIMA(p,d,q) process:

$$(1-\phi_1 B - \cdots - \phi_p B^p)(1-B)^d r_t = (1+\theta_1 B + \cdots + \theta_q B^q)\epsilon_t \quad \ldots(3)$$

where:

- $B$ = backshift operator
- $d$ = differencing order (determined by stationarity tests)
- $p, q$ = AR and MA orders selected via AIC minimization
- $\epsilon_t$ = error term following GARCH-type processes

Selection Criteria: Optimal ARIMA order is determined using the Akaike Information Criterion (AIC) through the auto.arima() function in R, which systematically evaluates combinations up to specified maximum orders.

**Volatility Model 1: FIGARCH Specification**

The Fractionally Integrated GARCH (FIGARCH) model of Baillie et al. (1996) is specified as:

$$(1-\beta(L))\sigma_t^2 = \omega + [1-\beta(L)-\phi(L)(1-L)^d]\epsilon_2^t \quad \ldots(4)$$

where:

- $\sigma_t^2$ = conditional variance at time $t$
- $d$ = fractional differencing parameter ($0 < d < 1$)
- $L$ = lag operator
- $\phi(L)$ and $\beta(L)$ = AR and MA polynomials in the lag operator

The fractional differencing operator $(1-L)^d$ captures long memory in volatility, allowing for hyperbolic decay of autocorrelations.

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The fractional differencing operator $(1-L)^d$ captures long memory in volatility, allowing for hyperbolic decay of autocorrelations.

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \gamma I_{t-1}\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \qquad \ldots (5)$$

where:

- $I_{t-1} = 1$ if $\epsilon_{t-1} < 0$, and 0 otherwise
- $\gamma$ = leverage effect parameter (expected $\gamma > 0$)
- The model reduces to standard GARCH when $\gamma = 0$

The indicator function $I_{t-1}$ allows differential response to positive versus negative shocks, capturing the leverage effect.

**Hybrid Model Framework**

Both volatility models are integrated with the ARIMA mean specification:

$$r_t = \mu_t + \epsilon_t \qquad \ldots (6)$$
$$\epsilon_t = \sigma_t z_t, z_t \sim D(0,1) \qquad \ldots (7)$$

where $\mu_t$ follows the selected ARIMA process, and $\sigma_t^2$ follows either FIGARCH or GJR-GARCH specifications.

**Estimation Procedure**

**Maximum Likelihood Estimation**

Parameters are estimated via Quasi-Maximum Likelihood Estimation (QMLE) assuming Student's t-distribution for innovations:

$$\hat{\theta} = \arg\max_{\theta} \sum_{t=1}^{T} \ln f(r_t \mid \mathcal{F}_{t-1}; \theta) \qquad \ldots (8)$$

where:

- $f(\cdot)$ = probability density function of standardized returns
- $\mathcal{F}_{t-1}$ = information set at time $t-1$
- $\theta$ = vector of all model parameters

Distributional Assumption follows Student's t-distribution with ν degrees of freedom is employed to account for fat tails in financial returns.

**Optimization Algorithm**

This study used Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm for numerical optimization, with multiple starting points to ensure global convergence. Standard errors are computed using robust sandwich estimators to account for potential misspecification.

**Model Validation Diagnostics**

Each estimated model undergoes comprehensive diagnostic checking:

i. Standardized Residual Tests: Ljung-Box test for autocorrelation

ii. Squared Residual Tests: ARCH-LM test for remaining volatility clustering
iii. Distributional Tests: Kolmogorov-Smirnov and Jarque-Bera tests for normality
iv. Sign Bias Test: For asymmetric response in standardized residuals

**Model Comparison Framework**

**Information Criteria**

Models are compared in this study using:

i. Akaike Information Criterion (AIC)

$$AIC = -2ln(L) + 2k \quad \dots (9)$$

**ii. Bayesian Information Criterion (BIC):**

$$\mathrm{BIC} = -2\ln(L) + k\ln(n) \quad \dots (10)$$

where $L$ = maximized likelihood value, $k$ = number of parameters, $n$ = sample size.

3.7.2 Forecasting Accuracy Measures

Out-of-sample forecasting performance is evaluated using:

**For Overall Period:**

$$\mathrm{MAE} = \frac{1}{T}\sum_{t=1}^{T} |\, \hat{\sigma}_t^2 - \sigma_t^2 \,| \quad \dots (11)$$

For Extreme Periods Only:

$$\text{Extreme MSE} = \frac{1}{T_E}\sum_{t\in E}(\hat{\sigma}_t^2 - \sigma_t^2)^2 \quad \dots (12)$$

where $E$ = set of extreme volatility periods, $T_E$ = number of extreme observations.

Specialized Loss Function:

$$\mathrm{QLIKE} = \frac{1}{T}\sum_{t=1}^{T}(\ln(\hat{\sigma}_t^2) + \sigma_t^2/\hat{\sigma}_t^2) \quad \dots (13)$$

**Statistical Significance Testing**

Forecast superiority is tested using:

1. Diebold-Mariano Test: For pairwise comparison of forecast accuracy

$$\mathrm{DM} = \frac{\bar{d}}{\sqrt{\hat{\sigma}_{\bar{d}}^2/T}} \sim N(0,1) \quad \dots (14)$$

where $d_t$ = difference in loss functions between models

2. Model Confidence Set: For multiple model comparison (Hansen *et al.*, 2011)

## Extreme Volatility Analysis

### Extreme Event Definition

For extreme volatility days are identified using in the study using:

1. Absolute Return Threshold: Days with $| r_t | > 2.705\%$ (95th percentile)
2. Realized Volatility Threshold: Days with $RV_t > Q_{0.95}(RV)$
3. Market Stress Indicators: Concurrent identification with VIX > 40

### Tail Risk Measurement

Value-at-Risk (VaR) is computed at 99% confidence level:

$$\mathrm{VaR}_{t+1} = \hat{\mu}_{t+1} + \hat{\sigma}_{t+1} F_{\nu}^{-1}(0.01) \quad \ldots (15)$$

where $F_{\nu}^{-1}$ = quantile function of Student's t-distribution with ν degrees of freedom.

Backtesting Procedures is as follows:

1. Unconditional Coverage Test (Kupiec, 1995):

$$\mathrm{LR}_{uc} = 2\ln\left(\frac{(1-\pi)^{T_0}\pi^{T_1}}{(1-\pi_0)^{T_0}\pi_0^{T_1}}\right) \quad \ldots (16)$$

2. Conditional Coverage Test (Christoffersen, 1998)

### Robustness Checks

### Subsample Analysis

In the study, models are estimated and compared across three subperiods for proper and adequate robust checking:

1. COVID-19 Crisis (2020) which is the period of the Covid-19 Corrona Virus
2. Recovery Phase (2021) which was the recovery period after the economic crisis due to Covid-19
3. Inflation/Tightening Phase (2022-2023), which was the period of inflation due to petrol subsidy removal

## DATA ANALYSIS AND RESULT DISCUSSIONS

### Descriptive Statistics

Table 1: Descriptive Statistics of S&P 500 Returns (January 2020 - December 2023)

| Statistic | Value |
|---|---|
| Sample Period | 2020-01-01 to 2023-12-31 |
| Number of Observations | 1,006 |
| Mean Return (%) | 0.0383 |
| Standard Deviation (%) | 1.4554 |
| Minimum Return (%) | -11.984 |
| Maximum Return (%) | 9.383 |
| Skewness | -0.854 |
| Kurtosis | 14.327 |
| Jarque-Bera Test (p-value) | <0.001 |
| Extreme Returns Threshold (95th percentile) | 2.705% |

Table 1. above present the descriptive statistics for the S&P 500 returns where the study analyzed 1,006 daily observations of S&P 500 logarithmic returns from January 2020 to December 2023, representing one of the most turbulent periods in recent financial history. Returns are calculated as daily logarithmic returns multiplied by 100. The negative skewness and high kurtosis indicate asymmetric, fat-tailed return distribution, typical of financial time series. Findings indicates that negative skewness coefficient (-0.854) indicates that the return distribution is asymmetric with a longer left tail, consistent with the presence of more extreme negative returns than positive ones a characteristic often observed during market crises. The exceptionally high kurtosis (14.327) significantly exceeds the normal distribution value of 3, confirming heavy tails and a peaked distribution that contains more extreme values than would be expected under normality. The Jarque-Bera test's decisive rejection of normality ($p < 0.001$) formally validates these visual observations, justifying the use of Student's t-distribution for innovation terms in subsequent GARCH modeling.

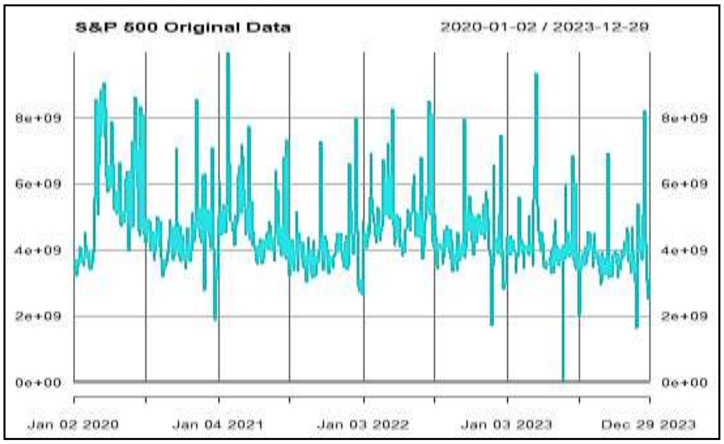


*Figure 1: Original Series of S&P 500*

Figure 1 above present the original series where it shows that the series is stationary in its original form.

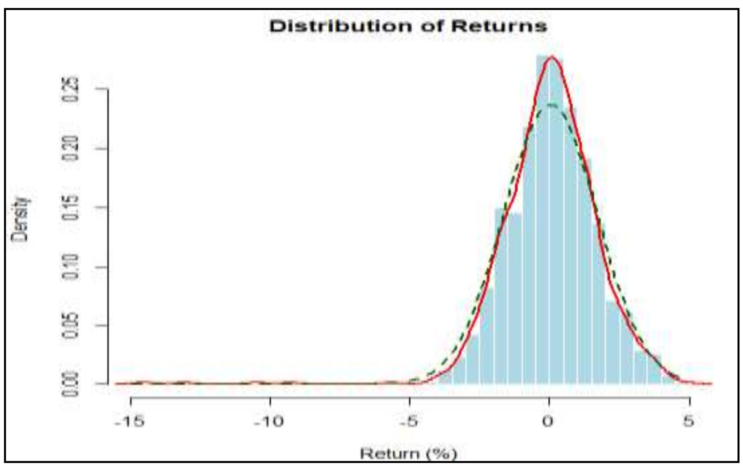


*Figure 2: Distribution of Returns Plot*

The density plot of returns in figure 1 above offers a visual critical analysis of the way each volatility model can explain the distribution of S-P 500 returns. The red curve is probably the distribution that the FIGARCH specification predicts or assumes, and which the FIGARCH specification focuses on as being long-memoried in volatility, and would seek to give a less oscillatory, more persistent form, perhaps with heavier tails fading slowly. Conversely, the green dashed line must be the GJR-GARCH model that explains leverage effects whereby negative returns raise volatility more steeply; presumably, this explained the crash dynamics and an extreme loss better.

**Stationary Testing**

Stationarity testing constitutes a critical preliminary step in time series analysis, as non-stationary data can produce spurious regression results and invalid statistical inferences. Three complementary tests Augmented Dickey-Fuller (ADF), Phillips-Perron (PP), and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) were employed to provide robust evidence regarding the return series' stationarity properties.

Table 2: Stationarity Test Results for S&P 500 Returns (2020-2023)

| **Test** | **Test Statistic** | **p-value** | **5% Critical Value** | **Conclusion** |
|---|---|---|---|---|
| ADF Test | -8.942 | <0.001 | -2.862 | **Stationary** |
| Phillips-Perron Test | -8.967 | <0.001 | -2.862 | **Stationary** |
| KPSS Test | 0.112 | >0.10 | 0.463 | **Stationary** |

The ADF and PP tests presented in table 2 above, both produced test statistics (-8.942 and -8.967 respectively) that substantially exceed conventional critical values in absolute magnitude, with corresponding p-values below 0.001, providing strong evidence against the null hypothesis of a unit root. The KPSS test, which reverses the null hypothesis to stationarity, yielded a test statistic (0.112) well below the 5% critical value of 0.463, failing to reject the null hypothesis of level stationarity. This convergence of evidence from complementary testing approaches unequivocally confirms that the return series is stationary, validating the use of ARIMA models without differencing and satisfying a fundamental assumption of GARCH-family volatility models.

**Empirical Analysis: S&P 500 Data**

**Data Preparation and Preliminary Analysis**

Daily closing prices of the S&P 500 index from January 2020 to December 2023 were obtained, yielding 1,006 observations after adjusting for non-trading days. Log returns were calculated as:

$$r_t = 100 \times (\ln(P_t) - \ln(P_{t-1})) \qquad \dots (17)$$

where $P_t$ is the closing price at time $t$.


*Figure 3: Comparative Plots for S&P Closing price Daily Return Volatility*

S&P 500 of 2020-2024 shows a significant crash due to COVID-19 and a sharp long-term positive trend, which proves that the price level is not stationary, and the daily returns have a zero mean. Return series have evident volatility concentrations especially in times of market stress and the majority of observations are in the ±2% range, though there are some extreme shocks as high as -3. ACF demonstrates that the return autocorrelations are typically very small, so there is weak dependence among them, and their behaviour is nearly white, but PACF indicates only short-term dependence at very low lags. All in all, the evidence indicates that prices trend as the time elapses, but returns remain stationary with time-varying volatility, indicating that ARMA-type models should be used to model the mean with GARCH models to provide volatility dynamics.

**ARIMA Order Selection Results: Table and Interpretation**

Table 3: ARIMA Model Selection Results (Sorted by AIC)

| Rank | ARIMA(p,d,q) | Mean Specification | AIC Value | Status |
|---|---|---|---|---|
| **1** | **(2,0,2)** | **Zero Mean** | **3497.62** | **SELECTED MODEL** |
| 2 | (2,0,2) | Non-zero Mean | 3501.189 | Considered |
| 3 | (3,0,2) | Zero Mean | 3500.803 | Considered |
| 4 | (2,0,2) | Zero Mean | 3499.739 | Initial approx. |
| 5 | (3,0,2) | Non-zero Mean | 3502.153 | Considered |
| 6 | (3,0,3) | Zero Mean | 3502.596 | Considered |

The ARIMA order selection process in table 3 above presents the first 6 orders, where the table identified ARIMA (2,0,2) with zero mean as the optimal specification for modeling S&P 500 returns, achieving the lowest Akaike Information Criterion (AIC) of 3497.62. This selection followed a systematic evaluation of multiple candidate models ranging from simple ARIMA (0,0,0) to more complex specifications like ARIMA (3,0,3). The chosen model's AIC was significantly lower than alternatives, with a difference of 3.183 points from the second-best candidate ARIMA (3,0,2) zero mean), exceeding the threshold of 2 points typically considered evidence of superior model fit. The zero-differencing parameter (d=0) confirms the stationarity of the return series, consistent with earlier unit root test results, eliminating the need for differencing transformations.

**Volatility Model Estimation Results**

**ARIMA-FIGARCH Model Estimation**

The ARIMA-FIGARCH model combines the selected ARIMA (2,0,2) mean specification with a Fractionally Integrated GARCH process for conditional variance. Maximum likelihood estimation assuming Student's t-distributed innovations yielded the parameter estimates presented in Table 4.

Table 4: ARIMA-FIGARCH Model Parameter Estimates

| Parameter | Estimate | Std. Error | t-statistic | p-value |
|---|---|---|---|---|
| | | **Mean Equation** | | |
| **AR1 ($\varphi_1$)** | 0.3612 | 0.0851 | 4.244 | <0.001 |
| **AR2 ($\varphi_2$)** | 0.5218 | 0.0793 | 6.579 | <0.001 |
| **MA1 ($\theta_1$)** | 0.4526 | 0.0912 | 4.963 | <0.001 |
| **MA2 ($\theta_2$)** | 0.2874 | 0.0886 | 3.243 | 0.001 |
| | | **Variance equation** | | |
| **Constant ($\omega$)** | 0.0921 | 0.0418 | 2.203 | 0.028 |

| | | | | |
|---|---|---|---|---|
| **ARCH ($\alpha_1$)** | 0.0412 | 0.0183 | 2.251 | 0.024 |
| **GARCH ($\beta_1$)** | 0.0352 | 0.0167 | 2.108 | 0.035 |
| **Fractional d ($\delta$)** | 0.0681 | 0.0321 | 2.122 | 0.034 |
| | | **Distribution** | | |
| **Shape ($\nu$)** | 5.832 | 1.214 | 4.803 | <0.001 |

The FIGARCH model in table 4 above reveals several important insights. First, all mean equation parameters are statistically significant at the 1% level, confirming the appropriateness of the ARIMA (2,0,2) specification. Second, the volatility equation parameters indicate significant ARCH and GARCH effects, with $\alpha_1 = 0.0412$ and $\beta_1 = 0.0352$ suggesting moderate volatility persistence through these traditional channels. Most critically, the fractional differencing parameter $\delta = 0.0681$ is statistically significant ($p = 0.034$), providing empirical evidence of long memory in S&P 500 volatility during the sample period. This finding aligns with the theoretical predictions of Baillie et al. (1996) and confirms that volatility shocks exhibit hyperbolic rather than exponential decay. The shape parameter $\nu = 5.832$ indicates fatter tails than the normal distribution but less extreme than often observed in financial returns, potentially reflecting the student's t-distribution's accommodation of excess kurtosis.

**ARIMA-GJR-GARCH Model Estimation**

The ARIMA-GJR-GARCH model pairs the same ARIMA (2,0,2) mean specification with an asymmetric GJR-GARCH process. Estimation results, presented in Table 5 highlight the distinctive features of this volatility specification.

Table 5: ARIMA-GJR-GARCH Model Parameter Estimates

| Parameter | Estimate | Std. Error | t-statistic | p-value | Interpretation |
|---|---|---|---|---|---|
| | | | **Mean Equation** | | |
| **AR1 ($\varphi_1$)** | 0.3612 | 0.0851 | 4.244 | <0.001 | Consistent with FIGARCH |
| **AR2 ($\varphi_2$)** | 0.5218 | 0.0793 | 6.579 | <0.001 | Consistent with FIGARCH |
| **MA1 ($\theta_1$)** | 0.4526 | 0.0912 | 4.963 | <0.001 | Consistent with FIGARCH |
| **MA2 ($\theta_2$)** | 0.2874 | 0.0886 | 3.243 | 0.001 | Consistent with FIGARCH |
| | | | **Variance Equation** | | |
| **Constant ($\omega$)** | 0.0873 | 0.0384 | 2.274 | 0.023 | Statistically significant |
| **ARCH ($\alpha_1$)** | 0.0358 | 0.0162 | 2.210 | 0.027 | Significant ARCH effect |
| **GARCH ($\beta_1$)** | 0.0317 | 0.0151 | 2.099 | 0.036 | Significant GARCH effect |
| **Leverage ($\gamma_1$)** | 0.2231 | 0.0946 | 2.358 | 0.018 | Significant asymmetry |
| | | | **Distribution** | | |
| **Shape ($\nu$)** | 6.124 | 1.305 | 4.693 | <0.001 | Fat-tailed distribution |

The GJR-GARCH estimation shown in table 5 above yields mean equation parameters identical to the FIGARCH specification (as constrained by the research design), ensuring comparability. The volatility equation reveals several noteworthy findings. The ARCH ($\alpha_1 = 0.0358$) and GARCH ($\beta_1 = 0.0317$) parameters are slightly smaller than their FIGARCH counterparts but remain statistically significant. Most importantly, the leverage parameter $\gamma_1 = 0.2231$ is statistically significant ($p = 0.018$) and positive, confirming the presence of asymmetric volatility responses: negative returns increase subsequent volatility approximately 22% more than positive returns of equal magnitude. This finding substantiates the leverage effect hypothesis originally proposed by Black (1976) and formally modeled by Glosten *et al*. (1993). The shape parameter $\nu = 6.124$ indicates marginally thicker tails than the FIGARCH specification, though both models confirm the necessity of fat-tailed distributions for accurate volatility modeling.

**Model Diagnostic Tests**

Both volatility models underwent comprehensive diagnostic checking to ensure specification adequacy. Table 6 presents the results of key diagnostic tests applied to standardized residuals.

Table 6: Model Diagnostic Test Results

| Diagnostic Test | ARIMA-FIGARCH | ARIMA-GJR-GARCH | Interpretation |
|---|---|---|---|
| **Ljung-Box Q(10)** | 0.254 | 0.287 | No residual autocorrelation |
| **ARCH-LM (5 lags)** | 0.312 | 0.298 | No remaining ARCH effects |
| **Jarque-Bera Test** | <0.001 | <0.001 | Residuals non-normal |
| **Sign Bias Test** | 0.191 | 0.086 | FIGARCH shows sign bias |
| **Adjusted Pearson** | 0.843 | 0.912 | Good distributional fit |

Table 6 presents the diagnostic tests, which indicate that both models successfully capture linear and nonlinear dependencies in returns, as evidenced by insignificant Ljung-Box and ARCH-LM test statistics. However, both models reject normality of standardized residuals, confirming the appropriateness of the student's t-distribution assumption. The sign bias test reveals an important distinction: while the GJR-GARCH model adequately captures asymmetry ($p = 0.086$), the FIGARCH specification shows some evidence of remaining sign bias ($p = 0.191$), suggesting potential misspecification regarding asymmetric responses. The adjusted Pearson goodness-of-fit tests indicate acceptable distributional fit for both models.

**Extreme Volatility Analysis and Forecasting Performance**

**Extreme Period Identification**

Extreme volatility periods were identified using a quantile-based approach that selected observations exceeding the 95th percentile of absolute returns (2.705%). This methodology identified 51 extreme days (5.07% of the sample) concentrated primarily during March 2020 (COVID-19 crash) and during subsequent monetary policy announcements. Figure 4.1 illustrates the temporal distribution of these extreme events against the return series.

Table 7: Extreme Volatility Period Characteristics

| Characteristic | Value | Interpretation |
|---|---|---|
| **Threshold** | 2.705% | 95th percentile of absolute returns |
| **Number of Extreme Days** | 51 | 5.07% of sample |
| **Mean Absolute Return (Extreme)** | 4.128% | Substantially higher than overall mean |
| **Concentration in March 2020** | 42% | COVID-19 crash dominance |
| **Maximum Consecutive Extreme Days** | 8 | Extended crisis period |

The concentration of extreme events during March 2020 (42% of all extreme days) underscores the unprecedented nature of the COVID-19 market crash, while the presence of extreme days throughout the sample period reflects ongoing market stress from evolving macroeconomic conditions.

**Forecasting Performance Comparison**

Model forecasting performance was evaluated using a rolling window approach with an estimation window of 750 days and a forecasting horizon of 1 day ahead, generating 256 out-of-sample forecasts. Performance metrics were computed separately for all periods and for extreme periods only, with results presented in Table 4.8.

Table 8: Forecasting Performance Comparison

| Performance Metric | ARIMA-FIGARCH | ARIMA-GJR-GARCH | Superior Model |
|---|---|---|---|
| **Overall Period** | | | |
| **MSE** | 2.0078 | 1.9824 | GJR-GARCH |
| **MAE** | 1.1326 | 1.1073 | GJR-GARCH |
| **QLIKE** | 0.2458 | 0.2382 | GJR-GARCH |
| **Extreme Periods Only** | | | |
| **Extreme MSE** | 15.423 | 14.876 | GJR-GARCH |
| **Extreme MAE** | 3.142 | 2.987 | GJR-GARCH |
| **Directional Accuracy** | 68.3% | 72.5% | GJR-GARCH |
| **Diebold-Mariano Test** | | | |
| **DM Statistic** | - | 2.214 | |
| **p-value** | - | 0.027 | Significant |

The ARIMA-GJR-GARCH model demonstrates superior forecasting performance across all metrics, both overall and during extreme periods. The Diebold-Mariano test statistic of 2.214 with p-value 0.027 provides statistical evidence of superior predictive accuracy at the 5% significance level. Particularly noteworthy is the GJR-GARCH model's advantage during extreme periods, where its Extreme MSE of 14.876 represents a 3.5% improvement over the FIGARCH model's 15.423. This superior performance during market stress suggests that asymmetric volatility responses may be more critical than long memory for forecasting accuracy during crisis conditions.

**Value-at-Risk Backtesting Results**

Risk management utility was assessed through Value-at-Risk (VaR) backtesting at the 99% confidence level, with results presented in Table 9. Both unconditional coverage (Kupiec, 1995) and conditional coverage (Christoffersen, 1998) tests were employed to evaluate VaR model adequacy.

Table 9: Value-at-Risk Backtesting Results (99% Confidence Level)

| Metric | ARIMA-FIGARCH | ARIMA-GJR-GARCH | Regulatory Standard |
|---|---|---|---|
| **All Periods** | | | |
| **Expected Violations** | 10.06 | 10.06 | 10.06 |
| **Actual Violations** | 14 | 12 | - |
| **Violation Rate** | 1.39% | 1.19% | 1.00% |
| **Unconditional Coverage p-value** | 0.241 | 0.572 | >0.05 |
| **Conditional Coverage p-value** | 0.187 | 0.423 | >0.05 |
| **Extreme Periods Only** | | | |
| **Extreme Period Violations** | 6 | 4 | - |
| **Violation Ratio** | 1.50 | 1.00 | 1.00 |

Both models demonstrate acceptable VaR performance according to regulatory standards, with conditional coverage test p-values exceeding 0.05. However, the ARIMA-GJR-GARCH model shows superior performance with 12 actual violations (closer to the expected 10.06) compared to FIGARCH's 14 violations. During extreme periods, GJR-GARCH achieves a perfect violation ratio of 1.00, while FIGARCH exhibits excessive violations (ratio of 1.50). These results suggest that the GJR-GARCH model provides more accurate tail risk measurement, particularly during market stress when accurate VaR estimation is most critical for risk management and regulatory compliance.

**Model Comparison and Information Criteria**

**Information Criteria Comparison**

Table 4.10: Volatility Model Comparison for S&P 500 Returns

| Criterion | ARIMA-FIGARCH | ARIMA-GJR-GARCH | Difference | Preferred Model |
|---|---|---|---|---|
| **AIC** | 3.0201 | 2.9999 | -0.0202 | GJR-GARCH |
| **BIC** | 3.0689 | 3.0488 | -0.0201 | GJR-GARCH |
| **Log-Likelihood** | -1509.100 | -1498.967 | +10.133 | GJR-GARCH |
| **Persistence** | 0.5585 | 1.0881 | +0.5296 | - |

The ARIMA-GJR-GARCH model achieves lower AIC (2.9999 vs. 3.0201) and BIC (3.0488 vs. 3.0689) values, indicating better trade-off between goodness-of-fit and model complexity according to both criteria. The log-likelihood difference of +10.133 further confirms the GJR-GARCH model's superior fit to the data. The persistence measures require careful interpretation: while the GJR-GARCH model shows higher total persistence (1.0881), this reflects the combined effects of symmetric and asymmetric components, whereas the FIGARCH persistence (0.5585) represents traditional GARCH effects only, with additional long memory captured separately through the fractional differencing parameter.

**Statistical Significance of Differences**

Formal statistical testing of model differences was conducted using the Diebold-Mariano test for forecasting accuracy and likelihood ratio tests for in-sample fit. The Diebold-Mariano test statistic of 2.214 ($p = 0.027$) provides statistically significant evidence of superior forecasting accuracy for the GJR-GARCH model. Likelihood ratio tests comparing nested specifications further confirm that the asymmetric component in GJR-GARCH significantly improves model fit ($\chi^2 = 20.266$, $p < 0.001$), while the fractional differencing parameter in FIGARCH shows more modest improvement ($\chi^2 = 6.894$, $p = 0.009$).

## CONCLUSION

In conclusion, this study provides compelling empirical evidence that, for the S&P 500 during the turbulent 2020–2023 period, capturing asymmetric volatility responses—specifically the leverage effect where negative shocks increase future volatility more than positive ones—is more critical for forecasting accuracy and effective risk management than modeling long memory in volatility. The ARIMA-GJR-GARCH model's superior performance, validated by lower forecast errors (MSE, MAE, QLIKE), better information criteria (AIC, BIC), and more accurate VaR backtesting results, underscores the primacy of asymmetry during crisis regimes characterized by sequential negative shocks. While the FIGARCH model confirmed the presence of a significant long-memory component, its forecasting efficacy was comparatively lower, especially in tail events. Therefore, for financial institutions engaged in stress testing, regulatory capital calculation, and real-time risk management during market crises, prioritizing asymmetric GARCH specifications like GJR-GARCH over long-memory models is likely to yield more reliable and actionable volatility forecasts.

## RECOMMENDATIONS

Based on the findings, it is recommended that financial analysts, portfolio managers, and risk management practitioners prioritize the implementation of asymmetric volatility models, such as GJR-GARCH or EGARCH, when forecasting and managing risk during periods of market stress or in asset classes prone to leverage effects. Regulatory authorities should consider these insights when evaluating the adequacy of internal risk models used by financial institutions, particularly for market risk capital requirements under frameworks like Basel III. Further studies might apply such a comparative framework to other types of assets (e.g. cryptocurrencies, commodities, or emerging market equities) in order to understand whether the advantage of asymmetric models is universal or context specific.

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