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Existence And Stability Of Common Fixed Points In Partially Ordered Sets

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ABSTRACT
In this paper, we have obtained existence and stability of common fixed points in partially ordered sets, where diagonally dominated system of equations were used in the first and second iteration, for effective and efficient results.
Keywords: Banach's fixed point theorem, monotone mappings

INTRODUCTION
The concept of fixed point in partially ordered sets has been a great and important area of research in Science, Engineering and Arts, most especially in the area of mathematical analysis. Banach's fixed point theorem also called contraction mapping theorem, which concerns with certain sets of mappings in a complete metric space (X,d) into itself $(F : X \to X)$. Aniki and Rauf (2022), established results on stability analysis of quadrupled fixed point theorems for mixed monotone mappings in partially ordered metric space. Also explained that the concept of stability is adapted from the iterative fixed point method, studied the stability of the coupled fixed point iterative procedures using some contractive conditions for which the existence of a unique coupled fixed point were established. Aniki (2022), justified existence and uniqueness of fixed point in partially ordered Cauchy space. Ekayanti, Muslikh, Adatul, Fitri and Marjono (2023), established result on fixed point in partially metric space. Khadija and Said (2020), proved some common fixed point theorems in partially ordered sets. Mohammed and Mohammed (2021), established results on existence and uniqueness of fixed point for monotone operators in partially ordered space and applications. Sani and Rauf (2019), established results on common fixed point theorem for five self mapping in a complete metric space. Aniki and Metiboba (2025), proved stability framework for Quintuple fixed point of mixed monotone operators with contractive conditions in Cauchy spaces. The study addresses the gap by developing a novel iterative procedures for quintuple fixed point, the methodology introduced a generalised contractive type mapping and used recursive inequalities alongside matrix-based comparison technique to establish convergence and stability criteria. Fernando, Policarpo, Jorge, Antonio, Serrano and Andreja (2025), proved results on levelled partially ordered sets. Juyu, Jianxiong, Dong, Laigui, Lanzhu and Mingxiang (2024), discussed an evaluation method of open-pit slope stability based on Poset theory. Carl, Bart, Steffen, Manuel and Reed (2009), established results on concept of stability and Posets.
As contained in Banach (1922) an effective and efficient result were obtained in the area of contraction mapping, equation with the condition stated as:
$d(Tx,Ty) \leq \alpha d(x,y)$, where $x,y \in X$, T is a self mapping on set X and $\alpha \in [0,1)$.
For instance it has been used to show the existence of solution of non-linear volterra integral equations, non-linear integro-differential equation in Banach spaces.

Two hundred and fifty (250) types of contractive definitions were provided by Rhodes (1977) and compared the relationship among them. Gregoria and Malansantos (2013), obtained results on fixed point theorems and stability of fixed point sets of multivalued mappings in partially ordered metric spaces and pre-order relations, which are obtained by imposing a distance condition to comparable elements of two non-empty, closed and bounded sets. John and Thomas (2024), established results on dynamic program on partially order sets. For more results on fixed point and partially ordered set, see: [Rhodes (1979); Davey and Priestley (2002); Berinde (2007); Bedinger and Reuter (2015); Alata (2016); Latpate and Dolhare (2016); Thangapandi, Antony and Maria-joseph (2016); Latpate and Dolhare (2017); Usamot (2017); Robab and Fariba (2019); Farajzadeh, Deltani and Wang (2021); Albin (2022); Saba (2022); Taeb, Buhlmann and Chandrasekaran (2024) and Fernando *et al*. (2025)].

There are a lot of results on fixed point theory, Poset and here we would consider existence and stability of common fixed points in partially ordered sets.

Definition 1.1. *Let X be a metric space and if F_1 and F_2 are any two maps. An element $a \in X$ is said to be a common fixed point of F_1 and F_2 if $F_1(a) = F_2(a)$.*

Definition 1.2. *Two elements a and b are said to be comparable if $a \leq b$ or $b \leq a$ otherwise they are incomparable. For example $\{x\}$ and $\{x,y,z\}$ are comparable while $\{x\}$ and $\{y\}$ are not.*

Definition 1.3. *A chain is a subset of a Poset that is totally ordered set. For example $\{\{\},\{x\},\{x,y,z\}\}$ is an chain.*

1 Preliminary Results

Some preliminary theorems, definitions and lemmas will be stated in this section in order to obtain results on existence and stability of common fixed points in partially ordered

sets.

Theorem 2.1. *(Banach Fixed Point): Every contraction mapping A defined on a complete metric space* R *has a unique fixed point.*

Theorem 2.2. *(Ekayanti et al. 2023): Let $(X,\leq)$ be a partially ordered sets and supposed that there exists a metric d in X such that (X,d) is a complete metric space. Let $F : X \to X$ be continuous and non-decreasing mapping such that there exists $k \in [0,1)$ with*

$$d(F(x),F(y)) \leq kd(x,y).$$

For all $x \geq y$. If there exists $x_0 \in X$ with $x_0 \in f(x_0)$ then F has a fixed point.

Definition 2.1. *Let* $(X,\leq)$ *be partially ordered set and* $F : X \to X$*. We say that F is monotone non-decreasing if* $x,y \in X, x \leq y \Rightarrow F(x) \leq F(y)$*. This coincides with the notion of non-decreasing function in the case where* $X = \mathrm{R}$ *and* $\leq$ *represents the usual total order in* R.

Definition 2.2. *If* $(X,\leq)$ *is a partially ordered set and* $F : X \to X$*, we say that F is monotone non-increasing if* $x,y \in X, x \leq y \Rightarrow F(x) \geq F(y)$.

Definition 2.3. *Fixed point iteration* $x_{n+1} = f(x_n)$ *is stable if a small iteration perturbation near the fixed point q, that is* $q = f(q)$*. Decay over time which the sequence converges to q. Stability is determine by the differential coefficient if* $/f'(q)/ \geq 1$ *the point is stable and if* $/f'(q)/ < 0$ *it is unstable.*

The theorem that follows immediately, expatiate more on existence of fixed point in partially ordered in partial metric space.

Theorem 2.3. *(Ekayanti et al.2023): Let* $(X,\leq)$ *be partially ordered set and suppose that there exist a metric d in X such that* (X,ϕ) *is a complete partial metric space. Let* $F : X \to X$ *be continuous and non decreasing mapping such that there exists* $k \in [0,1)$ *with* $x,y \in X$

$$\phi(F(x),F(y)) \leq k\phi(x,y) \tag{1}$$

for all $x \geq y$*. If there exists* $x_0 \in X$ *with* $x_0 \leq f(x_0)$*, then f has a fixed point, that is* x^**. further more* $\phi(x^*,x^*) = 0$.

Theorem 2.4. *(Khadija and Said 2020): Let* $(X,\leq)$ *be a partially ordered set such that every pair* $x,y \in X$ *has an upper and lower bound. Let* ϕ *be a partially metric on X such that* (X,ϕ) *is a complete partially metric space. Let* $f : X \to X$ *be a continuous and non-decreasing mapping. Suppose that the following conditions hold:*

(i) *If* $k \in (0,1)$ *the* $\phi(f(x),f(y) \leq k\phi(x,y)$*, for every* $x \geq y$

(ii) *There exists* $x_0 \in X$ *then* $x_0 \leq f(x_0)$ *or* $x_0 \geq f(x_0)$, *then* f *has a unique fixed point in* $x^* \in X$.

$$\text{Hence } \phi(x^*,x^*) = 0 \text{ and for each } x \in X,\ \lim_{n\to\infty}\phi(f^n(x),x^*) = \phi(x^*,x^*)$$

Theorem 2.5. *(Farajzadeh et al. 2021): Let* (X,d) *be a complete metric space and Let* $T : X \to X$ *be a contraction on* X. *Then* T *has a unique Fixed Point* $x \in X$ *such that* $T(x) = x$.

Theorem 2.6 (Latpate and Dolhare, 2017). *Let* (X,d) *be a complete metric space and let* A *be a non-empty closed subset of* X. *Suppose*

$P,Q,: A \to A$ *such that*

$$d(P_x,Q_y) \leq \frac{1}{2}(d(R_x,Q_y) + d(R_y,P_x) + d(S_x,R_y)) - \psi(d(R_x,Q_y),d(R_y,P_x)) \quad (2)$$

for any $(x,y) \in X{\times}X$ *where a function* $\psi : [0,\infty]^2 \to [0,\infty]$ *is continuous and* $\psi(x,y) = 0$ *iff* $x = y = 0$ *and* $R : A \to X$ *which satisfies the following conditions:*

(i) $P(A) \subseteq R(A)$, *and* $Q(A) \subseteq R(A)$

(ii) *the pair of mappings* (P,R) *and* (Q,R) *are weakly compatible.*

(iii) $R(A)$ *is a closed subset of* X.

Then P,R, and Q, have unique common fixed point.

Theorem (John and Thomas 2024): Let $(X,d,\leq)$ be a partially ordered complete metric space. let $F_1 :\to CB(X),(i = 0,1,2,...)$ be a sequence of multivalued mappings each satisfying all the conditions by $S(F_0),S(F_1),S(F_2),S(F_3),...$ the respective fixed point sets of $\Delta(F(X),F_0(X)) = 0$

Uniformly for all $x \in S$ and

(a) For all $x_0 \in S(F_0),\{x_0\} <_4 F_i(X_0)$ for $i = 1,2,3,...$

(b) For all $y_0 \in S(F_1),\{x_0\} <_4 F_0(y_0)$ for $i = 1,2,3,...$

Then $\lim_{n\to\infty}\Delta(S(F_n),S(F_0)) = 0$

Theorem (John and Thomas 2024): If (V,T) is finite and order stable:

(a) The Fundamental Approximate dynamic programming optimal (A.D.P) results holdand

(b) Howard policy iteration (H.P.I) converges infinitely many steps.

Theorem (John and Thomas 2024): Let (V,T) be order continuous and order stable. If

V countable Dedekind complete and (V,T) is bounded above then;

(a) The Fundamental A. D. P. optimal properties hold and

(b) Value function iteration (V. F. I.), Optimistic policy iteration (O. P. I) and H. P. I. allconverges.

Lemma (John and Thomas 2024): Let S be a self mapping on V with at most one fixed point in V . If either

(a) V is chain complete and S is order preserving or

(b) V is countably chain complete and S is order continuous, then S is order stable on V .

Lemma (Aniki and Metiboba 2025): Let $\{p_n\},\{q_n\},\{r_n\},\{s_n\}$ be sequence of nonnegative real numbers. Consider a matrix $A \in M_{(4,4)}$(R) with non negative elements, such that.

□ □ □ □ □ □ p_{n+1} p_n n_n
□ □ □ □ □ □

□□q_{n+1}□□ $\leq A$□□□q_n□□□+ □□□ϵ_n□□□
□ □
□□r_{n+1}□□ □□r_n□□ □□δ_n□□

□ □ □ □ □ □ s_{n+1} s_n
γ_n $n \geq 0$ with

(i) $lim_{n\to\infty} A_n = 0_4$

(ii) $\sum_{k=0}^{\infty} n_k < \infty, \sum_{k=0}^{\infty} \epsilon_k < \infty, \sum_{k=0}^{\infty} \delta_k < \infty$ and $\sum_{k=0}^{\infty} \gamma_k < \infty$

□ □ □□ □ □ □□ p_n 0 n_n 0
□ □ □□ □ □ □□
□□q_n□□ □□0□□ □□ϵ_n□□ □□0□□

If $lim_{n\to\infty}$ □ □ = □□, then □ □= □□,
□□r_n□□ □□0□□ □□δ_n□□ □□0□□

□ □ □□ □ □ □□ s_n 0 γ_n
0

Lemma (Aniki and Metiboba 2025): Let $\{a_n\}$ and $\{b_n\}$ be sequence of non-negative real numbers and let h be a constant such that $0 \leq h \leq 1$. Suppose the sequences satisfy the recursive inequality

$a_{n+1} \leq ha_n + \beta_n,; n \geq 0,$

If $lim_{n\to\infty}\beta_n = 0$, then it follows that $lim_{n\to\infty}a_n = 0$.

2 Main Results

3.1 Introduction

This section discussed the main results as follows:

Lemma: Suppose p satisfies identity property, V is asymtotically contractive maps on X and has a fixed point of q, only if q is unique and globally stable. Proof. Let q be a fixed point of V and $y \in X$, then $p(q,V'y) = p(V'q,V'y) \to 0$

Therefore, q is a globally stable fixed point of V . Also for fixed point s and q of V , we have

$0 \leq p(q,q)$

$\leq p(s,q)$

$= p(s,V'q)$

$= p(V's,V'q) \to 0$

Similarly, the above inequality is true if we replace $p(q,q)$ with $p(s,s)$ by identity property of p we have $q = s$.

Theorem 3.1. *Let* (X,d) *be a complete metric space and let* A *be a non-empty closed subset of* X. *Let* $P,Q,R,T,B : A \to A$ *such that* $R : A \to X$ *only if the common fixed point converges and satisfies the properties of partially ordered sets.*

Proof. Let x_0 be any arbitrary element of A as $P(A) \subseteq R(A), Q(A) \subseteq R(A), T(A) \subseteq R(A)$ and $B(A) \subseteq R(A)$.

$y_0 = Px_0 = Rx_1,\ y_1 =$

$Qx_1 = Rx_2,\ y_2 = Tx_2 =$

$Rx_3,\ y_3 = Bx_3 =$

$Rx_4, \cdots$ Let $\{x_n\}$ and

$\{y_n\}$ be two

sequences such that

$$y_{2n} = Px_{2n} = Rx_{2n+1},$$

$$y_{2n+1} = Qx_{2n+1} = Rx_{2n+2},$$

$$y_{2n+2} = Tx_{2n+2} = Rx_{2n+3},$$

$$y_{2n+3} = Bx_{2n+3} = Rx_{2n+4}, \cdots$$

First we prove that $d(y_n, y_{n+3}) \to 0$ as $n \to \infty$ then $d(y_{2n}, y_{2n+3}) \to 0$ as $n \to \infty$. Let $n = 2k$ implies,

$$d(y_n, y_{n+3}) = d(y_{2k}, y_{2k+3}) \tag{3}$$

Also

$$d(y_{2n+3}, y_{2n}) = d(Bx_{4k+3}, Px_{4k}) \tag{4}$$

$$\leq \frac{1}{4}(d(Rx_{4k}, Bx_{4k+3}) + d(Rx_{4k+1}, Tx_{4k+2}) +$$

$$d(Rx_{4k+2}, Qx_{4k+1}) + d(Rx_{4k+3}, Px_{4k}) + d(Sx_{4k}, Rx_{4k+3}))$$

$$- \psi(d(Rx_{4k}, Bx_{4k+3}), d(Rx_{4k+1}, Tx_{4k+2}), d(Rx_{4k+2}, Qx_{4k+1}),\ d(Rx_{4k+3}, Px_{4k})) \tag{5}$$

$$= \frac{1}{4}(d(y_{4k-1}, y_{4k+3}) + d(y_{4k}, y_{4k}) + d(y_{4k}, y_{4k}) + d(y_{4k}, y_{4k}) +$$

$$d(y_{4k}, y_{4k}))$$

$$- \psi(d(y_{4k-1}, y_{4k+3}), d(y_{4k}, y_{4k}), d(y_{4k}, y_{4k}), d(y_{4k}, y_{4k})) \tag{6}$$

$$\leq \frac{1}{4}(d(y_{4k-1}, y_{4k+3}))$$

$$\frac{1}{4}(d(y_{4k-1}, y_{4k}) + d(y_{4k}, y_{4k+1}) + d(y_{4k+1}, y_{4k+2}) + d(y_{4k+2}, y_{4k+3}) \tag{7}$$

by expanding for $n = 2k + 1$, similarly we can show that

$d(y_{2n}, y_{2n+3}) \to 0$ as $n \to \infty$

$d(y_{2n},y_{2n+3}) = d(y_{4k+4},y_{4k+1})$

$$\leq d(y_{4k+4},y_{4k+3}) \leq d(y_{4k+3},y_{4k+2}) \quad (8)$$

$$\leq d(y_{4k+2},y_{4k+1}) \leq d(y_{4k+1},y_{4k})$$

Let

$$\leq d(y_{4k+4},y_{4k+1}) \leq d(y_{2n+3},y_{2n})$$

$d(y_{2n+3},y_{2n}) = d(y_{4k+4},y_{4k+1})$ is a non-increasing sequence of non-negative real numbers and hence it is convergent.

Let $L = \lim_{n\to\infty}(y_{4k+4},y_{4k+1})$ from (8) we have $d(y_{4k+4}, y_{4k+1}) \leq \frac{1}{4}(d(y_{4k}, y_{4k+4}))$ and by contraction mapping,

$$d(y_{4k+4}, y_{4k+1}) \leq \frac{1}{4}(d(y_{4k-1}, y_{4k+1})+(d(y_{4k+1}, y_{4k+2})+(d(y_{4k+2}, y_{4k+3}+(d(y_{4k+3}, y_{4k+4})) \quad (9)$$

Letting $k \to \infty$, then,

$$\lim_{k\to\infty} d(y_{4k+2}, y_{4k+3}) \leq \frac{1}{4} \lim_{k\to\infty} d(y_{4k}, y_{4k+1}) \leq \lim_{k\to\infty} d(y_{4k+1}, y_{4k+2})$$

$$\leq \lim_{k\to\infty} d(y_{4k+2}, y_{4k+3} \leq \lim_{k\to\infty} d(y_{4k+3}, y_{4k+4}) \quad (10)$$

$$L \leq \frac{1}{4} \lim_{k\to\infty} d(y_{4k}, y_{4k+4}) \leq L \leq L \leq L \quad (11)$$

$$\lim_{k\to\infty} d(y_{4k}, y_{4k+4}) = 4L \quad (12)$$

$$d(y_{2n+3}, y_{2n}) = d(Px_{4k+1}, Bx_{4k+4})$$

$$\leq \frac{1}{4}(d(y_{4k-1}, y_{4k}) + d(y_{4k}, y_{4k+1}) + d(y_{4k+1}, y_{4k+2}) + d(y_{4k+2}, y_{4k+3})+$$

$$d(y_{4k+3}, y_{4k+4})) - \psi(d(y_{4k-1}, y_{4k}), d(y_{4k}, y_{4k+1})$$

$$, d(y_{4k+1}, y_{4k+2}), d(y_{4k+2}, y_{4k+3}))$$

consider

(13)

(14)

Letting $k \to \infty$ and since ψ is given to be continuous, therefore we obtained

$$L \leq \frac{1}{4}(4L) - \psi(4L, 0) \tag{15}$$

this gives

$$\psi(4L,0) = 0 \tag{16}$$

By definition of $\psi(x,y) = 0$ if $x = y = 0$

$4L = 0, \Rightarrow L = 0$

$$L = \lim_{k\to\infty} (y_{4k+1}, y_{4k+4}) = 0 \tag{17}$$

Now our claim is that $\{y_{4k}\}$ is a Cauchy sequence and from (17) it implies

$$d(y_{4k+2}, y_{4k+4}) \leq d(y_{4k+1}, y_{4k+2}) \tag{18}$$

To prove $\{y_{4k+1}\}$ is Cauchy sequence, we only prove that the subsequence $\{y_{4k}\}$ is a Cauchy. Suppose that $\{y_{4k}\}$ is not Cauchy sequence there exist $\delta>0$ for which we can find subsequences $\{y_{4n(k)}\}$ and $\{y_{4m(k)}\}$ of $\{y_{4k}\}$. such that n_k is the least index for which $n_k > m_k > k$ and

$$d(y_{4m(k)}, y_{4n(k)}) \geq \delta \tag{19}$$

This gives

$$d(y_{4m(k)}, y_{4n(k)-4}) < \delta \tag{20}$$

By contraction mapping,

$$\begin{aligned} \delta \leq d(y_{4m(k)}, y_{4n(k)}) \leq d(y_{4m(k)}, y_{m(k)-4}) + d(y_{4m(k)-4}, y_{4m(k)-3}) + \\ d(y_{4m(k)-3}, y_{4m(k)-2}) + d(y_{4m(k)-2}, y_{4m(k)-1}) + d(y_{4m(k)-1}, y_{4n(k)}) \end{aligned} \tag{21}$$

Now as k$\to \infty$ and from (19) we obtain

$$\lim d(y_{4m(k)}, y_{4n(k)}) = \delta \; k\to\infty \tag{22}$$

By triangle inequality, we have

$$| d(y_{4m(k)}, y_{4n(k)-1}) - d(y_{4m(k)}, y_{4n(k)}) | \leq d(y_{4n(k)-1}, y_{4n(k)}) \tag{23}$$

Also

$$|\, d(y_{4n(k)},y_{4m(k)-1}) - d(y_{4n(k)},y_{4m(k)}) \,| \leq d(y_{4m(k)},y_{4m(k)-1}) \tag{24}$$

Again

$$|\, d(y_{4n(k)},y_{4m(k)-1}) - d(y_{4n(k)},y_{4m(k)-2}) \,| \leq d(y_{4m(k)-1},y_{4m(k)-2}) \tag{25}$$

$$|\, d(y_{4m(k)-1},y_{4m(k)-2}) - d(y_{4m(k)-1},y_{4m(k)-3}) \,| \leq d(y_{4m(k)-2},y_{4m(k)-3}) \tag{26}$$

$$|\, d(y_{4m(k)-2},y_{4m(k)-3}) - d(y_{4m(k)-2},y_{m(k)-4}) \,| \leq d(y_{4m(k)-3},y_{4m(k)-4}) \tag{27}$$

(19),(22),(23),(25),(26) and (27) we have

$$\lim_{k\to\infty} d(y_{4n(k)-1},y_{4n(k)}) \leq \lim_{k\to\infty} d(y_{4m(k)-1},y_{4m(k)})$$
$$= \lim_{k\to\infty} d(y_{4m(k)-1},y_{4m(k)-2}) = \lim_{k\to\infty} d(y_{4m(k)-2},y_{4m(k)-3}) = \lim_{k\to\infty} d(y_{4m(k)-3},y_{4m(k)-4}) \tag{28}$$

Hence

$$d(y_{4m(k)-3},y_{4n(k)}) = d(Px_{4n(k)},Bx_{4m(k)-3}) \tag{29}$$

$$\begin{aligned}&\leq \frac{1}{4}(d(Rx_{4n(k)},Bx_{4m(k)-3}) + d(Rx_{4m(k)-1},Tx_{4n(k)-2})\\ &+ d(Rx_{4m(k)-2},Qx_{4n(k)-1}) + d(Rx_{4m(k)-3},Px_{4n(k)}) + d(Sx_{4n(k)},Rx_{4m(k)-3}))\\ &- \psi(d(Rx_{4n(k)},Bx_{4m(k)-3}),d(Rx_{4m(k)-1},Tx_{4n(k)-2}),\\ &d(Rx_{4m(k)-2},Qx_{4n(k)-1}),d(Rx_{4m(k)-3},Px_{4n(k)}))\end{aligned} \tag{30}$$

$$\frac{1}{4}(d(y_{4m(k)-1},y_{4m(k)}) + d(y_{4m(k)},y_{4m(k)+3})) \tag{31}$$

$$\begin{aligned}&\frac{1}{4}(d(y_{4n(k)-1},y_{4m(k)-1}) + d(y_{4m(k)-1},y_{4m(k)}) + d(y_{4m(k)},y_{4m(k)+1})\\ &+ d(y_{4m(k)+1},y_{4m(k)+2}) + d(y_{4m(k)+2},y_{4m(k)+3}))\\ &- \psi(d(y_{4n(k)-1},y_{4m(k)-1}),d(y_{4m(k)-1},y_{4m(k)}),d(y_{4m(k)},y_{4m(k)+1}),d(y_{4m(k)+1},y_{4m(k)+2}))\end{aligned}$$

Letting $k \to \infty$ from (15) and ψ is continuous, therefore we have

$$\delta \leq \frac{1}{4}(\delta+\delta+\delta+\delta) - \psi(\delta,\delta,\delta,\delta) \tag{32}$$

This gives $\psi(\delta,\delta,\delta,\delta) = 0$ by assumption of $\psi(x,y) = 0$ if $x = y = 0$ therefore $\delta = 0$ but $\delta > 0$ there is contradiction, therefore $\{y_n\}$ is a Cauchy sequence to prove P,Q,R,T,B have a common fixed point given (X,d) is complete and $\{y_n\}$ is a Cauchy sequence. There exists $p \in X$ such that $\lim_{n\to\infty} y_n = p$ as A is closed. So there is u$\in A$ such that

$p = Ru$ for every $n \in \mathbb{N}$.

$d(Pu, y_{2n+3}) = d(Pu, Bx_{2n+3})$

since

$$d(y_{2n}, y_{2n+3}) = d(Pu, Bx_{2n+3}) = d(y_{4k}, y_{4k+3}) \tag{33}$$

$$\leq \frac{1}{4}\big(d(Ru, Bx_{4k+3}) + d(Rx_{4k+2}, Tu) + d(Rx_{4k+1}, Qu) + d(Rx_{4k}, Pu) + d(Su, Rx_{4k})\big) - \psi\big(d(Ru, Bx_{4k+3}), d(Rx_{4k+2}, Tu), d(Rx_{4k+1}, Qu), d(Rx_{4k}, Pu)\big), \tag{34}$$

$$= \frac{1}{4}\big(d(P, y_{4k+3}) + d(y_{4k+2}, Tu) + d(y_{4k+1}, Qu) + d(y_{4k}, Pu) + d(Su, y_{4k})\big) - \psi\big(d(Ru, Bx_{4k+3}), d(y_{4k+2}, Tu), d(y_{4k+1}, Qu), d(y_{4k}, Pu)\big) \tag{35}$$

The corresponding Rx_{4k} and y_{4k} in (34) and (35) are obtained from iteration function sequence from the beginning of our prove. When $k \to \infty$

$$d(Pu, P) \leq \frac{1}{4}\big(d(P, Bu) + d(P, Tu) + d(P, Qu) + d(Pu, P) + d(Su, P)\big) - \psi\big(d(P, Bu), d(P, Tu), d(P, Qu), d(Pu, P)\big) \tag{36}$$

and hence

$$\psi(0, d(P, Pu)) \leq -\frac{1}{4}\big(d(P, Bu) + d(P, Tu) + d(P, Qu) + d(Su, P)\big) \leq 0 \tag{37}$$

therefore $d(P, Pu) = 0$ since $Pu = P$ similarly, we can show that $Bu = p$, $Tu = p$, $Qu = p$, and $Su = p$

therefore

$$Bu = Tu = Qu = Ru = Pu = p \tag{38}$$

To show that the common fixed point satisfies the properties of partially ordered set as:

(i) Supposed that $p = a$ and for all $a \in X$, we know that reflexive means ${}_a\mathrm{R}_a$, also called reflexive if $f(a) \leq f(a)$ which is true by considering the property (P1) of partially ordered sets $(X, \leq)$.

(ii) Let $a, b \in X$ and the relation $\leq$ be partially ordered set, then $f(a) \leq f(b)$ and $f(b) \leq f(a)$ which implies that $a \leq b$ and $b \leq a$ by property two of Partially ordered sets. Similarly, the converse is also true. Therefore, $_aR_b$ and bRa hold.

(iii) Let $a,b,c \in X$, then a relation $\leq$ is called partially ordered sets if $f(a) \leq f(b)$ implies that $f(b) \leq f(c)$ then $f(a) \leq f(c)$ $_aR_b$ and $_bR_c$ then $_aR_c$. Hence the proof.

Theorem 3.2. *Let (X,d) be a complete metric space and let A be a non-empty closed subset of X. Let $P,Q,R,T,B : A \to A$ such that $R : A \to X$ only if the common fixed point of P,R,Q,T and B, is stable.*

Proof From the proof of theorem 3.1, it could be shown that if a is a diagonally dominated linear system defined by

$$X_{(r+1)} = b' - B'X_r$$

$$\begin{bmatrix} 6 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 7 \end{bmatrix}$$

$6x + y + z = 8$ x

$+ 4y + z = 6$ $x +$

$y + 5z = 7$

Above equations are diagonally dominated.

$x = 8 - y - z/6$ y

$= 6 - x - z/4$ $z =$

$7 - x - y/5$

$$x^0 = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix}, x^1 = \begin{bmatrix} 1.3333 \\ 1.1667 \\ 1.1250 \end{bmatrix}, x^2 = \begin{bmatrix} 0.9514 \\ 0.9809 \\ 1.0135 \end{bmatrix},$$

$$x^3 = \begin{pmatrix} 1.0009 \\ 0.9964 \\ 1.0005 \end{pmatrix},\ x^4 = \begin{pmatrix} 1.0005 \\ 0.9998 \\ 1.0000 \end{pmatrix},\ x^5 = \begin{pmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \end{pmatrix},$$

$$x^6 = \begin{pmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \end{pmatrix},\ x^7 = \begin{pmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \end{pmatrix},\ x^8 = \begin{pmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \end{pmatrix},$$

Therefore, the required numerical solution is given by $x = 1$, $y = 1$ and $z = 1$.

The equation b can be expressed as:

$$X_{r+1} = b' - (L' X_{r+1} + U' X_r)$$

$$X_{1}^{r+1} = b' - \left(\sum_{j=1}^{i-1} L_{ij} x_j^{r+1} + \sum_{j=i+1}^{n} U_{ij} x_j^{r}\right)$$

$$X_{1}^{r+1} = b' - \left(\sum_{j=1}^{i-1} (a_{ij}/a_{ii}) x_j^{r+1} + \sum_{j=i+1}^{n} U_{ij} x_j^{r}\right)$$

$$\begin{pmatrix} 6 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 7 \end{pmatrix}$$

$$x^0 = \begin{pmatrix} 0.0000 \\ 0.0000 \\ 0.0000 \end{pmatrix},\ x^1 = \begin{pmatrix} 1.3333 \\ 1.1670 \\ 0.9000 \end{pmatrix},\ x^2 = \begin{pmatrix} 0.9170 \\ 1.0460 \\ 1.0007 \end{pmatrix},$$

0.9910 1.0000 1.0000

$$x^{3} = \begin{bmatrix} 1.0010 \\ 1.0002 \\ 1.0000 \end{bmatrix}, \quad x^{4} = \begin{bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix}, \quad x^{5} = \begin{bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix},$$

$$x^{6} = \begin{bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix}, \quad x^{7} = \begin{bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix}, \quad x^{8} = \begin{bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix}$$

Therefore, the required numerical solution is given by $x = 1$, $y = 1$ and $z = 1$.

Table1: First Iteration Table

s/n	x	y	z
0	0.0000	0.0000	0.0000
1	1.3333	1.1667	1.1250
2	0.9514	0.9809	1.0135
3	1.0009	0.9964	1.0005
4	1.0005	0.9998	1.0000
5	1.0000	1.0000	1.0000
6	1.0000	1.0000	1.0000
7	1.0000	1.0000	1.0000
8	1.0000	1.0000	1.0000

Table2: Second Iteration Table

s/n	x	y	z

0	0.0000	0.0000	0.0000
1	1.3333	1.1670	0.9000
2	0.9170	1.0460	1.0007
3	0.9910	1.0010	1.0002
4	1.0000	1.0000	1.0000
5	1.0000	1.0000	1.0000
6	1.0000	1.0000	1.0000
7	1.0000	1.0000	1.0000
8	1.0000	1.0000	1.0000

Corollary 3.3 Suppose $K = \{P,Q,R,T,B\}$; $H = \{C,D,E,F,J\}$ such that $R : A \to X$; $D : A \to X$ only if the common fixed point converges and satisfies the properties of partially ordered sets.

Proof: The proof follows directly from theorem 3.1

Corollary 3.4 If $K = \{P,Q,R,T,B\}$; $H = \{C,D,E,F,J\}$ such that $R : A \to X$; $D : A \to X$ only if K and H are stable.

Proof: The proof is similar to the proof of theorem 3.2.

CONCLUSION

In this research work, we have obtained existence and stability of common fixed points in partially ordered sets, where diagonally dominated system of equations were used in the first and second iteration Tables above (Table 1 and Table 2), in order to obtained effective and efficient results as required.

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