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Existence of An Associator Subloop For Moufang Loops Of Odd Order p^2q^4 With $N \neq \emptyset$

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ABSTRACT
In this paper, we show the existence of an associator subloop, L_a, of order q^2 for a nonassociative Moufang loop, L, of odd order p^2q^4 with nontrivial nucleus; where $3 \leq p < q$ are primes and $q \not\equiv 1(\mathrm{mod}\ p)$ such that all proper subloops and proper quotient loops of L are associative.
Keywords: Moufang loop, order, nonassociative, subloops, associator.

1 INTRODUCTION
A Moufang loop $\langle L,\cdot\rangle$ is defined as a loop that satisfies the identity: $xy \cdot zx = (x \cdot yz)x$ for all $x, y, z \in L$. Extensive work has been done in answering the question "Which Moufang loop is associative given it order n, where n is a positive integer?", but yet this question is still open. Chee and Rajah (2014) had shown that all Moufang loops of odd order pq^4 are associative if p and q are odd primes with $p < q$ and $q \not\equiv 1(mod\ p)$. In sequential order, the next order to investigate are Moufang loops of order p^2q^4.

As part of efforts to prove that Moufang loops of this order are associative or otherwise we prove the existence of the associator subloop, L_a, of order q^2 for this class of Moufang loops with nontrivial nucleus; where $3 \leq p < q$ are primes and $q \not\equiv 1(mod\ p)$ such that all proper subloops and proper quotient loops of L are associative.

2 Definitions
Definition 2.1 A Moufang loop is a loop $\langle L,\cdot\rangle$ that satisfies the identity $(x \cdot y) \cdot (z \cdot x) = (x \cdot (y \cdot z)) \cdot x$ for all $x, y, z \in L$. (Henceforth, for the sake of brevity, we shall define L as a finite Moufang loop $\langle L,\cdot\rangle$. Also, we shall write $(x \cdot y) \cdot z$ simply as $xy \cdot z$, $(x \cdot (y \cdot z)) \cdot x$ as $(x \cdot yz)x$, e.t.c.)

Definition 2.2 The associator subloop of L, denoted by L_a, is the subloop generated by all the associators $(x, y, z) \in L$ where $(x, y, z) = (x \cdot yz)^{-1}(xy \cdot z)$. L_a is also defined as
$L_a = (L, L, L) = \langle (l_1, l_2, l_3) | l_i \in L \rangle$ where $(l_1 l_2 \cdot l_3) = (l_1 \cdot l_2 l_3)(l_1, l_2, l_3)$. Indeed L is associative if and only if $L_a = \{1\}$.

Definition 2.3 Let K be a subloop of L. K is said to be a normal subloop of L, denoted $K \lhd L$, if $K\theta = K$ for all $\theta \epsilon I(L)$.

Definition 2.4 Let K be a normal subloop of L.
(a) K is said to be a proper normal subloop of L if $K \neq L$ and $K \neq \{1\}$.
(b) L/K is a proper quotient loop of L if $K \neq \{1\}$.

Definition 2.5 Suppose K is a normal subloop of L.
(a) K is said to be a minimal normal subloop of L if K is non-trivial and contains no proper non-trivial subloop which is normal in L.
(b) K is said to be a maximal normal subloop of L if K is not a proper subloop of every other proper normal subloop of L.

Definition 2.6 The nucleus of L is the subloop generated by all $n \in L$ such that $(n, x, y,) = (x, n, y) = (x, y, n) = 1$ for any $x, y \in L$ and it is denoted as $N = N(L)$.

Definition 2.7 Let K be a subloop of L and π a set of primes.
(a) A positive integer n is a π -number if every prime divisor of n lies in π.
(b) For each positive integer n, we let n_π be the largest π -number that divides n.
(c) K is a π -loop if the order of every element of K is a π -number.
(d) K is a Hall π -subloop of L if $|K| = |L|_\pi$.
(e) K is a Sylow p -subloop of L if K is a Hall π -subloop of L such tha π contains only a single prime p.

Definition 2.8 (m, n) is defined as the greatest common divisor of the integers m and n.

3 Known Results

Presented below are some known results, useful and used in this work.

Lemma 3.1 Suppose L is a Moufang loop, then the nucleus $N = N(L)$ is a normal subloop of L. (Bruck, 1971,Theorem 2.1, p.114).

Lemma 3.2 Suppose L is of odd order and K a normal Hall subloop of L. Let $K = \langle x \rangle L_a$ for some $x \in K \backslash L_a$ and $L_a \subset N$. Then $K \subset N$. (Rajah and Chee, 2011, Lemma 3.17, p.373).

Lemma 3.3 Let L be of odd order, K a subloop of L, and π a set of primes. Then
(a) L is solvable. (Glauberman, 1968, Theorem 16, p.413).
(b) L contains a Hall π -subloop. (Glauberman, 1968, Theorem 12, p.409).

Lemma 3.4 $|K|$ divides $|L|$ for every K a subloop of L. (Grishkov and Zavarnitsine, 2005, Lagrange's theorem, p.42).

Lemma 3.5 Any Moufang loop L is a group if $|L|$ is any of the following orders:
(a) p, p^2, p^3 *or* pq; for p and q distinct primes. (Chein, 1974, Corollary 4 and Proposition 3, p.35).
(b) pqr or p^2q; for p, q and r odd primes with $p < q < r$. (Purtill, 1988, Theorem 3.1, p.124 and Theorem 3.3, p.126).
(c) pq^2; for p and q distinct odd primes. (Leong and Rajah, 1995, Theorem, p.269).

(d) $p^{\alpha}q_1^{\beta_1}q_2^{\beta_2}\dots q_n^{\beta_n}$; for p and q_i primes with $p < q_1 < \cdots < q_n$, and β_i, with $\alpha \leq 3$, or $\alpha \leq 4$ when $p > 3$. (Leong and Rajah, 1997, Theorem 1, p.482 and Theorem 2, p.483).

(e) $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n}q^3$; for p_i and q distinct odd primes with $q \not\equiv 1 (\mathrm{mod}\, p_i)$, $q > p_i$ and $1 \leq \alpha_i \leq 2$. (Rajah and Chee, 2011, Theorem 4.2, p.970).

(f) pq^4, for p and q odd primes with $p < q$, and $q \not\equiv 1 (\mathrm{mod}\, p)$. (Chee and Rajah, 2014, Theorem 4.8, p.433).

Lemma 3.6 Let $|L|$ be odd and every proper subloop of L be a group. If there exists a minimal normal Sylow subloop in L, then L is a group. (Leong and Rajah, 1995, Lemma 2, p.268).

Lemma 3.7 Suppose L is of odd order such that every proper subloop and proper quotient loop of L is a group. Let Q be a Hall subloop of L such that $(|L_a|, |Q|) = 1$, and $Q \lhd L_a Q$. Then L is a group. (Leong and Rajah, 1996, Lemma 3, p.564).

Lemma 3.8 Let L be a finite Moufang loop and $R \lhd L$. If the quotient L/R is a group then $L_a \subset R$. (Leong and Rajah, 1996, Lemma 1(a), p.563).

Lemma 3.9 Suppose L is a nonassociative Moufang loop of odd order such that all proper quotient loops of L are groups. Then L_a is a minimal normal subloop of L (Leong and Rajah, 1997, Lemma 1(a), p.478); and is an elementary abelian group. (Glauberman, 1968, Theorem 7, p.402).

Lemma 3.10 Suppose $|L| = p^{\alpha}m$ where p is the smallest prime dividing $|L|$ with $(p, m) = 1$, $|L|$ is odd and $\alpha \in \{1,2\}$. Then there exists a subloop M of order m normal in L. (Ademola and Rajah, 2016, Lemma 4.2, p.1400).

Lemma 3.11 Suppose L is a Moufang loop of order $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n}q$, where $p_1, p_2, \dots, p_n$ and q are odd primes with $p_1 < p_2 < \cdots < p_n < q$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{Z}^+$, $q \not\equiv 1 (\mathrm{mod}\, p_i)$ for all i. Then there exists a normal subloop of order $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n}$ in L. (Rajah and Chee, 2011, Lemma 4.1, p.1362).

4 New Results

Theorem 4.1 *Let L be a nonassociative Moufang loop of odd order p^2q^4 with a nontrivial nucleus; where p and q are primes with $3 \leq p < q$ and $q \not\equiv 1 (mod\ p)$, such that all proper subloops and proper quotient loops of L are associative. Then: (a) $L_a \subset N$ and (b) $|L_a| = q^2$.*

Proof:

(a) Given that N is nontrivial and $N \lhd L$, by Lemma (3.1), then L/N is a group by the hypothesis. And $L_a \subset N$ by Lemma (3.8).

(b) By Lemma (3.3)(a) L is solvable. Also by Lemma (3.4), every proper subloop and quotient loop of L is of order $p^{\alpha}q^{\beta}$, for $0 \leq \alpha \leq 2$ and $0 \leq \beta \leq 4$ with $\alpha + \beta < 6$. These proper subloops and quotient loops of L are associative by Lemma (3.5)(a,b,c,d,e and f). Thus, by Lemma (3.9), L_a is a minimal normal subloop of L and an elementary abelian group. Furthermore, since L is not associative, $|L_a| \neq p^2$ or q^4 by Lemma (3.6). Hence,

$$|L_a| = p, q, q^2 \text{ or } q^3 \quad (1)$$

By Lemma (3.10), there exists

$Q \lhd L$ and $|Q| = q^4$ (2)

This implies $\left|\frac{L}{Q}\right| = p^2$, which is a group by Lemma (3.5)(a), and by Lemma (3.8) $L_a \subset Q$. Thus $|L_a|$ divides $|Q|$. Then by (1) $|L_a| = q, q^2$ or q^3.

Assume $|L_a| = q$: By Lemma (3)(b), there exists P a Hall subloop of order p^2 in L. Since $L_a \lhd L$ by Lemma (3.9), it implies that $L_aP < L$. So $|L_aP| = \frac{|L_a||P|}{|L_a \cap P|} = p^2q$. P is a Sylow p-subloop of L_aP, by Lemma (3.11) $P \lhd L_aP$. Also $(|P|, |L_a|) = (p^2, q) = 1$, so by Lemma (3.7), L is a group. This is a contradiction to our hypothesis.

Now assume $|L_a| = q^3$. By (a), $L_a \subset N$. Now $|Q| = q^4$, where Q is the normal subloop of order q^4 by (2), $|L_a| = q^3$ and $L_a \lhd Q$, so $Q = \langle x \rangle L_a$ for some $x \in Q \backslash L_a$. Also $\left(|Q|, \left|\frac{L}{Q}\right|\right) = (q^4, p^2) = 1$. Thus by Lemma (3.2), $Q \subset N$. But Q is a Hall subloop of L, so by Lemma (3.7), L is a group; this too is a contradiction to our hypothesis. Hence $|L_a| \neq q^3$.
Therefore $|L_a| = q^2$.

CONCLUSION

We have in this paper successfully established the existence of an associator subloop, L_a, of order q^2 for Moufang loops of odd order p^2q^4 with nontrivial nucleus; where $3 \leq p < q$ are primes and $q \not\equiv 1 (\bmod p)$ such that all proper subloops and proper quotient loops of L are associative.

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