International Journal of Innovative Mathematics, Statistics & Energy Policies
14(3):133-140, July-Sept, 2026

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DOI:10.5281/zenodo.22200958

A New Logarithmic Lomax–Inverse Exponential Model: Properties, And Application In Modeling Reliability Data

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ABSTRACT
The main motivation for this paper is to develop a more flexible distribution for the lifetime applications called Logarithm-Lomax inverse exponential (Log-LIE) distribution. The Log-LIE density has exponential, negatively-skewed, positively-skewed and symmetrical shapes. The Log-LIE distribution has increasing, increasing and decreasing and modified bathtub hazard rate function. Some properties of the Log-LIE model were established and this include moment, moment generating function, entropy and order statistics. The flexible nature of the hazard rate function of Log-LIE distribution helps to serve as a much better alternative model to the other current models for modelling real data in reliability and survival analysis. The Log-LIE distribution provides better fits than other sub-models.
Keywords: Hazard rate function, moments, Renyi entropy, order statistics.

1.0 INTRODUCTION

A flexible model for the analysis of real-world data set(s) is often very attractive to the researchers. Inverse exponential is of interest due to its flexibility and simplicity. In recent decades, many continuous univariate distributions have been proposed and studied for various data sets from reliability, insurance, climatology, medical sciences and other areas do not follow these distributions. Therefore, modified, extended and generalized distributions and their applications to problems in these areas is a clear reality in the present world. The modified, extended and generalized distributions are obtained by the introduction of one or more parameters to the baseline distribution which can be done through transformation or addition of parameter(s). These new developed distributions provide better fit to data than the sub and competing models.

A fundamental contribution in this direction by developing a flexible generator that used in baseline transformation was made by Eugene and Famoye (2002), who developed the Beta-normal distribution using the logit transformation of a Beta-distributed variable. Marshall and Olkin (1997) introduced a flexible approach in adding parameters to existing distributions. Since then, numerous generators have been developed, including the Burr XII-G family by Alizadeh et al. (2017), the Weibull-G family by Bourguignon et al. (2014), the Gompertz-G family by Alizadeh et al. (2017), and the odd Chen-G family by Al-Shomrani and Al-Arfaj (2021), among others Within the Lomax-based family, several notable extensions have been proposed. Cordeiro and Ortega (2015) introduced the Gamma–Lomax distribution as a flexible three-parameter model. Alabid and Hurairah (2019) generalized the power distribution using the quadratic rank transmutation map, while Sapkota eta l. (2023) developed a Lomax-G framework based on a Beta-generated transformation. More recently, Logarithmic Lomax family was introduced by Huda (2026).

2.0 logarithmic Lomax–Inverted Weibull Model

a random variable follows the Inverted Weibull distribution if the distribution (CDF) and the density (PDF) functions are respectively given as

$$J(x;\theta) = \exp(-\theta x^{-1}) \ , x > 0; \theta > 0 \qquad (1)$$

and

$$j(x;\theta) = \theta x^{-2}\exp(-\theta x^{-1}) , x > 0; \theta > 0 \qquad (2)$$

Where δ is scale parameter. On the basis of the work of Huda (2026) the PDF of the Logarithmic family of distribution is given as

$$f(x;\lambda,\vartheta) = \frac{\vartheta\theta^{\vartheta} g(x)}{G(x)}[\delta - logG(x)]^{-(\vartheta+1)} \qquad (3)$$

Inserting (1) and (2) into (3), we derive the PDF Log-LIE

$$f(x;\lambda,\vartheta) = \vartheta\delta^{\vartheta}\theta x^{-2}[\delta + \theta x^{-1}]^{-(\vartheta+1)} \qquad (4)$$

The associated CDF to (4) is given as

$$F(x;\lambda,\vartheta) = [\delta + \theta x^{-1}]^{-(\vartheta+1)} \qquad (5)$$

Where δ, and ϑ are the positive shape parameters and θ is a scale parameter. An expression for the survival and the hazard functions is respectively given as

$$S(x) = 1 - F(x) = 1 - [\delta + \theta x^{-1}]^{-\vartheta} \qquad (6)$$

And

$$h(x) = \frac{\vartheta\delta^{\vartheta}\theta x^{-2}[\delta + \theta x^{-1}]^{-(\vartheta+1)}}{1 - [\delta + \theta x^{-1}]^{-\vartheta}} \qquad (7)$$

the graphs of the distribution, density, survival, and the hazard functions is given in Figures 1 and 2 for various values of the parameters.

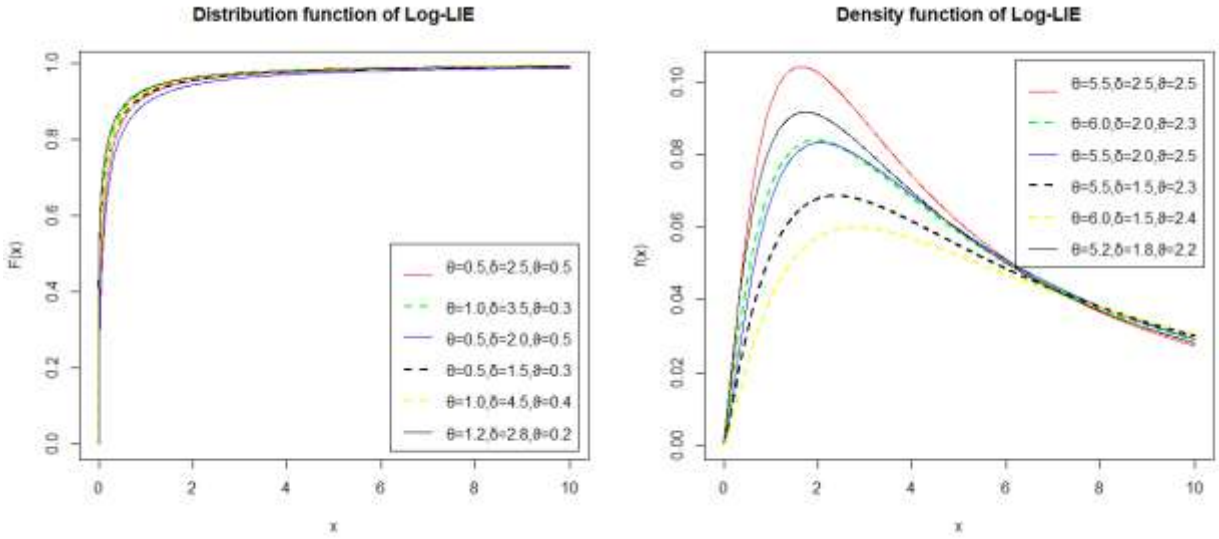

Figure 1.0 Graphs of the CDF and the PDF of the Logarithmic-LIE distribution

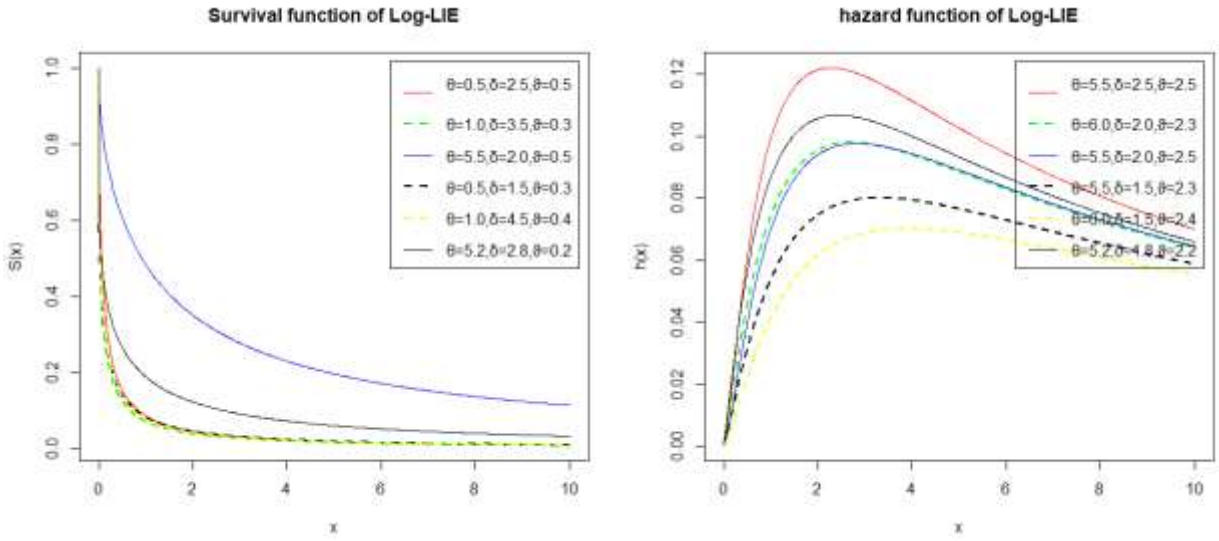

Figure 2.0 Graphs of the $S(x)$ and the $R(x)$ of the Log-LIE distribution

2.1 Quantile function for the Log-LIE Model

The quantile function, which is used to generate random samples from the Log-LIE distribution is derived via obtaining the solution to the nonlinear equation given by

$$1 - [\delta + \theta x^{-1}]^{-\vartheta} = u$$

Finally, we have

$$x_u = \left\{ {}^{1}/_{\theta} \left(1-u\right)^{-{}^{1}/_{\vartheta}} - \delta \right\}^{-1}. \qquad (8)$$

3.0 Moments

Let X follows the Log-LIE distribution then, the r^{th} ordinary moment of X can be derived from (11) as follows:

$$E(X^r) = \mu'_r = \int_{-\infty}^{\infty} x^r f(x)dx \qquad (10)$$

Inserting (4) in (10) we have

$$\mu'_r = \vartheta\delta^{\vartheta}\theta \int_{-\infty}^{\infty} x^{r-2}\left[\delta + \theta x^{-1}\right]^{-(\vartheta+1)} dx, \qquad (11)$$

By letting, $y = \frac{a}{x}$, $x = \frac{a}{y}$, $dx = \frac{-a}{y^2}dy$ and substitute in (13), we obtain

$$\mu'_r = \vartheta\delta^{\vartheta}\theta \int_{0}^{\infty} t^{-r}\left[\delta + y\right]^{-(\vartheta+1)} dy, \qquad (12)$$

Using the beta-integral

$$\int_{0}^{\infty} t^{u-1}\left[b + t\right]^{-m} dt = b^{u-m}B(u, m-u) \qquad (13)$$

gives

$$\mu'_r = \left(\frac{\theta}{\delta}\right)^r \vartheta B(1-r, \vartheta + r). \qquad (14)$$

In terms of the beta function, we have

$$\mu'_r = \left(\frac{\theta}{\delta}\right)^r \frac{\Gamma(1-r)\Gamma(\vartheta + r)}{\Gamma(\vartheta)}. \qquad (15)$$

3.1 The moment generating function and characteristic function.

Using the $e^{tX} = \sum_{r=0}^{\infty} t^r x^r / r!$, we can derive an expression for the moment generating function based on the r^{th} moment of the Log-LIE distribution as

$$M_X(t) = E(e^{tX}) = \int_{0}^{\infty} e^{tX} f_{log-LIE}(x)dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_{0}^{\infty} x^r f(x)dx \qquad (16)$$

Finally, we have

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left(\frac{\theta}{\delta}\right)^r \vartheta B(1-r, \vartheta + r). \qquad (17)$$

3.3 Rényi Entropy

The Rényi entropy (Rényi; 1961) for order $v > 0, v \neq 1$ is

$$I_R(\varphi) = \frac{1}{1-v} log\left(\int_0^\infty f(x)^v \, dx\right), \tag{18}$$

Plugging (4) in (19), we have

$$I_R(\varphi) = \frac{1}{1-v} log\left(\left(\vartheta\delta^\vartheta\theta\right)^v \int_0^\infty ([\delta + \theta x^{-1}]^{-(c+1)})^v \, dx\right), \tag{19}$$

Using the transformation, $y = \theta x^{-1}$, finally we obtain

$$I_R(\varphi) = \frac{1}{1-v} log(\vartheta^v \theta^{1-v} \delta^{v-1} B[2v-1, 1+v(\delta-1)]) \tag{20}$$

3.4 Order Statistics for the Log-LIE

For a random sample $X_1, X_2, \ldots, X_n$, let $X_1 < X_2 < \cdots < X_n$ denote the order statistics. The PDF of the k^{th} order statistic is

$$f_{X_k}(x) = \frac{n!}{(k-1)!(n-i)} \{F(x)\}^{k-1} \{1 - F(x)\}^{n-k} f(x) \tag{21}$$

Inserting (4) and (5),

$$f_{X_k}(x) = \frac{n!\,\delta^\vartheta\theta}{(k-1)!\,(n-i)} \left\{[\delta + \theta x^{-1}]^{-(\vartheta+1)}\right\}^{k-1} \left\{1 - [\delta + \theta x^{-1}]^{-(\vartheta+1)}\right\}^{n-k} x^{-2} \times [\delta + \theta x^{-1}]^{-(\vartheta+1)} \tag{22}$$

For the smallest order, $k = 1$, we have

$$f_{X_1}(x) = \frac{n\delta^\vartheta\theta}{(k-1)!} x^{-2} \left\{1 - [\delta + \theta x^{-1}]^{-(\vartheta+1)}\right\}^{n-1} [\delta + \theta x^{-1}]^{-(\vartheta+1)} \tag{23}$$

And the largest order $k = n$, we obtain

$$f_{X_k}(x) = n\delta^\vartheta\theta \left\{[\delta + \theta x^{-1}]^{-(\vartheta+1)}\right\}^{n-1} x^{-2} [\delta + \theta x^{-1}]^{-(\vartheta+1)} \tag{24}$$

4.0 Maximum likelihood estimation

Maximum likelihood estimation is defined as a method where the values of the unknown parameters of a distribution are estimated by maximizing the likelihood function of the distribution. Mathematically, given a sample of data $X_N = x_{N1}, x_{N2}, x_{N3}, \ldots, x$ and a probability distribution $f(X_N/\Omega_N)$ where Ω_N are unknown parameters, the likelihood function is represented as

$$L(\Omega_N, x_N) = \prod_{i=1}^{n} f(x_N) \qquad (25)$$

And the log-likelihood function

$$l(\Omega_N, x_N) = logL(\Omega_N, x_N) = log\left\{\prod_{i=1}^{n} f(x_N)\right\} \qquad (26)$$

Inserting (6) in (34) we have

$$l = nlog\left(\vartheta\delta^{\vartheta}\theta\right) - 2\sum_{i=1}^{n}\log(x_i) - \vartheta\sum_{i=1}^{n} log\left(\delta + \theta x^{-1}\right) \qquad (27)$$

4.1 Real life Application

The real data represents Aircraft windshield failure times obtained from [28], represents n = 84 recorded failure times of aircraft windshields: 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663. The fits of the Log-LIE distribution were compared to that of the Frechet, Inverted Weibull and Weibull using the following measures of goodness-of-fit statistics: Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC) and the Crammer-Von misses (CVM). The best model corresponds to the model showing the smallest AIC, BIC, and CVM, except for the highest PV. The aircraft data is slightly positively skewed as shown in Figure (3a), and mesokurtic. It also exhibits an increasing failure rate and bimodal as demonstrated in Figures (3b) and (3c) respectively. The exploratory data analysis is given in

Table 1. Summary Statistics of the Aircraft data

n	$minimum$	q_1	$Mean$	$Median$	q_3	$Maximum$	$Kurtosis$	$skewness$
84	0.04	1.84	2.56	2.35	3.39	4.66	2.35	0.10

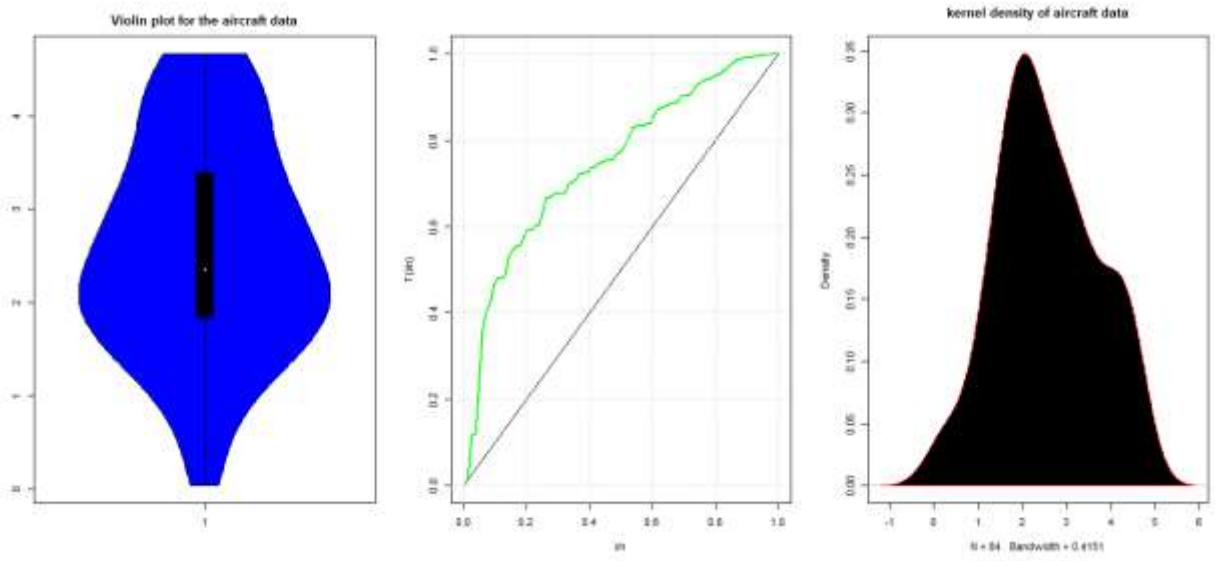

Table 2. Measures of the goodness of fit for the aircraft data

$Model$	θ	δ	ϑ	$-l$	AIC	$CAIC$	BIC	CVM
$Log-LIE$	0.3794 (0.1046)	0.7528 (0.0432)	4.0089 (1.2904)	174.47	362.95	363.25	370.24	0.84
F	1.3646 (0.1516)	0.8387 (0.0521)	– (–)	194.54	393.07	393.22	397.94	1.71
W	0.0823 (0.0227)	2.3747 (0.2095)	– (–)	130.05	264.11	264.26	268.97	0.06
IE	1.2273 (0.1339)	– (–)	– (–)	199.36	400.72	400.77	403.15	2.02

5.0 Conclusion

We have developed the Log-LIE distribution along with its properties such as: descriptive measures based on the quantiles, moments, stress-strength reliability and multi component stress-strength reliability model. The Log-LIE distribution is characterized via different techniques. Maximum Likelihood estimates are computed. Goodness of fit shows that Log-LIE distribution is a better fit. Applications of the Log-LIE to aircraft data are presented to demonstrate its significance and flexibility. We have shown that the Log-LIE distribution is empirically better aircraft data.

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