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Crank-Nicolson Error in High-Volatility Option Pricing: Evidence from Nigerian Equities

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ABSTRACT
Crank-Nicolson (CN) finite difference schemes are widely used to solve the Black-Scholes (BS) partial differential equation, but their accuracy depends critically on grid resolution relative to volatility. We compare CN approximations to exact Black-Scholes prices for European calls and puts across volatilities from 25% to 65%, using parameters representative of Nigerian cement equities. Results show CN systematically underprices both instruments, with absolute error reaching 0.77 for calls and 1.49 for puts, corresponding to 3.68% and 3.36% relative error at 65% volatility. Quantile functions reveal that 95th percentile stress-case errors are 0.68 for calls and 1.31 for puts. We formalize these findings in four theorems proving that CN error is non-negative, increasing in σ^2 , and bounded below by $C\sigma^2\Delta S^2$ for calls and puts. The results imply that risk engines using CN with standard grids will understate VaR by 3% + during Nigerian volatility clusters. Grid refinement or adaptive methods are required for regulatory compliance. Finally, the results presented here have weighty financial remunerations which are revealing to option traders and society at large.
Keywords: Crank-Nicolson, Black-Scholes, Volatility clustering, Numerical error, Option pricing and Finite difference methods.

1.1 INTRODUCTION

The Black-Scholes-Merton framework provides closed form solutions for European options under constant volatility and lognormal asset dynamics. In practice, most banks and trading desks use finite difference methods like Crank-Nicolson to price options, especially when dealing with early exercise, barriers, or local/stochastic volatility. Crank-Nicolson is preferred because it is unconditionally stable and second-order accurate in space and time. However, the Black-Scholes equation has been used broadly. Also, many researchers have used Black-Scholes Equation in different approaches. For instance; [1] considered Black-Scholes partial differential equation on stock market prices for both analytic and numerical. In another scenario focused on the Black-Scholes [18]terminal value problem and provided its solutions through the Laplace transform. More so, [6] proposed a framework based on the celebrated transform of Mellin type (MT) for the analytic solution of the Black-Scholes-Merton European Power Put Option Model (BSMEPPOM) on Dividend Yield (DY) with Modified-Log-Power Payoff Function (MLPPF) under the geometric Brownian motion. In the same vein, [4] analyzed BS formula for the valuation of European options; hermit polynomials were applied. They concluded that BS formula can easily be achieved devoid of the use of partial differential equation. In the work of [17], time varying factor were incorporated in the explicit formula for different aspect of options with the aim of providing exact solution for dividend paying equity of option. In considering

the stability of stock market price of stochastic model, [12] applied Crank-Nicolson numerical scheme to BS model. The results showed stock prices being stable and its increasing rate of stock shares was obtained. Not quite long, [11] investigated the variation of stock market price using BS PDE. The convergence to equilibrium of growth rate and sufficient conditions for stability was achieved. However, [3] studied the perception of European option which is geared towards valuation of financial assets, the application of share prices of Fidelity and Access banks which gave closed form prices of call options.

Nevertheless, [2] modified Black-Scholes model to assume a probability which measures risk-free interest rate of the underlying asset for Call and Put options. The Black-Scholes exact values and Modified Black-Scholes values were obtained and compared to close form prices. Lots of scholars has written extensively to mention but a few: [2],[7-8], [10],[13], [15-16] and[18] etc

Nonetheless, while CN is theoretically accurate, its performance deteriorates when the diffusion coefficient $\frac{1}{2}\sigma^2 S^2$ becomes large. This occurs in high-volatility regimes. Empirical studies on Nigerian equities , particularly cement stocks like Dangote Cement and BUA Cement ,show pronounced volatility clustering, with annualized volatility frequently exceeding 50% during FX crises and policy shocks. Standard CN grids calibrated for 20-30% volatility produce systematic pricing errors when applied in these regimes. The errors are not symmetric: they manifest as underpricing of both calls and puts, with error magnitude increasing quadratically in σ .

This paper quantifies the pricing bias of Crank-Nicolson relative to Black-Scholes exact values for European call and put options across volatility levels 25% to 65%. We construct empirical quantile functions to assess distributional errors, and we derive theoretical bounds on CN error as a function of volatility. The aim is to demonstrate that CN grids suitable for developed markets fail under Nigerian market conditions, and to provide theorems that formalize the error behavior. Existing literature focuses on CN stability and convergence rates for fixed σ . Few studies examine the practical impact of volatility clustering on numerical error, especially in frontier markets. No prior work has linked CN discretization error to VaR misstatement for NGX-listed single stocks. This study fills that gap by using real volatility ranges observed in Nigerian cement into financial metrics like CDF shifts and quantile errors; proving theorems that bound CN bias for calls and puts separately.

This paper is arranged as follows: Section 2.1 is material and methods, Section 3.1, Results and discussion while the paper is concluded in 4.1

2.1 MATERIAL AND METHODS

Here we present some basic definitions that cut across the area of study:

Definition 1: Stochastic Processes : Stochastic process: A stochastic process $X(t)$ is a relation of random variables $\{X_t(\gamma), t \in T, \gamma \in \Omega\}$, i.e, for each t in the index set T, $X(t)$ is a random variable. Now we understand t as time and call $X(t)$ the state of the procedure at time t. In view of the fact that a stochastic process is a relation of random variables, its requirement is similar to that for random vectors.

It can also be seen as a statistical event that involves time in accordance to probabilistic laws. Mathematically, a stochastic process may be defined as a collection of random variables which are ordered in time and defines at a set of time points which may be continuous or discrete.

Definition 2: Random Walk: There are different methods to which we can state a stochastic process. Then relating the process in terms of movement of a particle which moves in discrete steps with probabilities from a point $x = a$ to a point $= b$. A random walk is a stochastic sequence $\{S_n\}$ with $\{S_0 = 0\}$, defined by

$S_n = \sum_{k=1}^{n} X_k$ (1)

Where X_k are independent and identically distributed random variables.

Definition 3: Brownian Motion: The Brownian motion W_t or Wiener process is a basic process that serves as part of many different processes.
It refers either to: (i) The physical event that small particles immersed in a fluid go randomly; (ii)The Mathematical tool used to explain those random activities. It is the scaling bound of random walk in one aspect as the time steps to zero that is the quantity of steps becomes large.
Mathematically, Brownian motion $W_t, 0 \leq t \leq s$ is a set of random variables, each value of the real t in interval,
This set has the subsequent characteristics:
W_t is continuous function of the parameter t ,with $W_0 = 0$
For each $s < t$, W_t is normally distributed with expected value 0 and independent of each other. $W_t - W_s \sim N(0, t-s)$
Conversely , In order to use a Brownian Motion as a Model for stock prices, S at time t we prefer the stock returns over the period to have a constant drift, μ and volatility , σ.
Definition 4: Stochastic Differential Equation (SDE):A stochastic differential equation is a differential equation with stochastic term. let $(\Omega, \mathrm{F}, \wp)$ be a probability space with filtration $\{f_t\}_t \geq 0$ and $W(t) = (W_1(t), W_2(t), ..., W_m(t))^T$, $t \geq 0$ an m-dimensional Brownian motion on the given probability space. We have a stochastic differential equation in coefficient functions of f and g

as follows:

$$dX(t) = f(t, X(t))dt + g(t, X(t))dZ(t),\ 0 \leq t \leq T \tag{2}$$

$X(0) = x_o$

Where $T > 0$, x_0 is an n-dimensional random variable and coefficient functions are in the form $f: [0,T] \times R^n$ and $g: [0,T] \times R^n \to R^{n \times n}$. Sde can also be written in the form of integral as follows:

$$X(t) = x_0 + \int_0^t f(S, X(S))dS + \int_0^t g(S, X(S))dZ(S) \tag{3}$$

Where dX, dZ are terms known as stochastic differentials The R^n is a valued stochastic process $X(t)$.

However, let $S(t)$ be the price of some risky asset at time t, and μ ,an expected rate of returns on the stock and dt as a relative change during the trading days such that the stock follows a random walk which is govern by a stochastic differential equation.

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW_t \tag{4}$$

Suppose we have an option whose $V(S,t)$ depends only on S and t . Assuming also that the asset price is perturbed by a small change dS, then the function V will also change. Using Ito's lemma

$$dV = \sigma S \frac{\partial V}{\partial S} dx + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt \tag{5}$$

According to [3] and [7] the value of one portfolio having one stock can be expressed with the function $V(S,t)$.

$$\pi = V - \Delta S \tag{6}$$

The change in the portfolio at time dt in (4) is given by

$$d\pi = dV - \Delta S \tag{7}$$

Putting (4),(5) into (7),we find that π follows random walk, given by.

$$d\pi = \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dx + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt$$

To eliminate the random component in this random walk, let

$$\Delta = \frac{\partial V}{\partial S} \tag{8}$$

Note that Δ is the value of $\frac{\partial V}{\partial S}$ at the start of the time step dt. This results in a portfolio whose increment is wholly deterministic so that

$$d\pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \tag{9}$$

Now that the portfolio is riskless it should earn riskless return. The change in the portfolio at time dt becomes (after substituting (4) and (5) into (7) and dividing through by dt)

$$d\pi = r\pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$, and

$$r(V - \Delta S) dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

which implies that

$$rV - rS\frac{\partial V}{\partial S} = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

This gives the solution

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{10}$$

Where r represents interest rate, σ represents volatility of the underlying assets and t represents time of maturity. The details of the above option model can be expressly found in the following papers: [19-20] etc.

$$\left.\begin{array}{l} V(S,t) \to \infty \ as \ S \to \infty \ \ on \ [0,T) \\ V(S,t) \to 0 \ as \ S \to 0 \ \ on \ [0,T) \end{array}\right\} \tag{11}$$

And final time condition given by :

$$V(S_T,T) = (S_T - k)^+ = f(S_T) \ on \ [0,\infty]. \tag{12}$$

To eliminate the price process in (10-12) slightly gives the Black-Scholes analytic formula for the prices of European call option is given as follows

$$\left.\begin{array}{l} C = SN(d_1) - Ke^{-rt}N(d_2) \\ d_1 = \dfrac{\ln\left(\dfrac{S}{K}\right) + \left(r + \dfrac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \\ d_2 = d_1 - \sigma\sqrt{\tau} \end{array}\right\} \tag{13}$$

where C is Price of a call option, S is price of underlying asset, K is the strike price, r is the riskless rate , τ is time to maturity, σ^2 is variance of underlying asset, σ is standard deviation of the underlying asset (generally referred to as volatility), and N is the cumulative normal distribution. Similarly Black-Scholes analytic formula for the prices of European Put option is given as follows

$$\left.\begin{array}{l} P = Ke^{-rt}N(d_1) - SN(d_2) \\ \dfrac{ln\left(\dfrac{S}{K}\right) + \left(\dfrac{r+\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2 = d_1 - \sigma\sqrt{T} \end{array}\right\} \quad (14)$$

Where P is the price of put option and the meaning of other parameters remain the same as in [1-2] and [13-14].

2.2 The Numerical Scheme and Analysis

Here to implement Crank-Nicolson approximation scheme on Black-Scholes partial differential equation, there must be a price time mesh in order to enhance efficiency as solution exits, the vertical axis in the mesh denotes the stock prices, while the horizontal axis denotes time. Therefore, every grid point in the mesh denotes a horizontal index i and a vertical index j such that every point in the mesh is the option price for a distinct time and a distinct stock price. At every time in the mesh $j\Delta s$ is equivalent to the stock price, and $i\Delta t$ is equivalent to the time. There exist boundary conditions which help in the numerical calculations; by means of the pay-off function. The maturity period, $t = T$ and the option are well computed for all the different initial stock prices using a boundary conditions for uniqueness of solution. To get the prices at $t = 0$, the model solves backwards for every time step from $t = T$,[20-21]etc.

2.2.1 Formulation of the Scheme

One of the normal ways of approximating the solution of partial differential equations is applying Crank-Nicolson finite difference method which we shall use our proposed model to transform into the scheme. Recall that the Black-Scholes partial differential equation (10). Let a function $V(S,t)$ in two dimensional grid points, that is to say i and j stands for the index for stock price, S and time, t respectively. The function $V(S,t) = V_i^j$ can be stated as follows in the subsequent difference scheme.

$$Z_i^j = \frac{1}{2}\sigma^2 S^2 DSS + rS_i DS - rV_i^j \quad (15)$$

where

$S = i\Delta s$, for $0 \le i \le m$, $t = j\Delta t$ for $0 \le j \le i$

$$DSS = \frac{V^j{}_{i+1} - 2V^j{}_i + V^j{}_{i-1}}{\Delta^2} \quad (16)$$

$$DS = \frac{V^j{}_{i+1} - V^j{}_{i-1}}{2\Delta S} \quad (17)$$

Taking forward difference and backward difference approximations respectively yields implicit and explicit schemes given below.

If we use a forward difference approximation to the time partial derivative we obtain explicit scheme

$$\frac{V_i^{j+1}-V_i^j}{\Delta t}+Z_i^j=0 \tag{18}$$

and similarly we obtain the implicit scheme

$$\frac{V_i^{j+1}-V_i^j}{\Delta t}+Z_i^{j+1}=0 \tag{19}$$

The averages of equations (18) and (19) yields Crank-Nicolson method of approximation

$$\frac{V_i^{j+1}-V_i^j}{\Delta t}+\frac{1}{2}\left(Z_i^j+Z_j^{j+1}\right)=0 \tag{20}$$

From equation (20)

$$V_i^j-\frac{\Delta t}{2}Z_i^j=V_i^{j+1}+\frac{\Delta t}{2}Z_i^{j+1} \tag{21}$$

$$\frac{\Delta t}{2}Z_i^j=V_i^{j+1}-V_i^j+\frac{\Delta t}{2}Z_i^{j+1}$$

$$\therefore Z_i^j=\frac{2}{\Delta t}\left(u_i^{j+1}-u_i^j\right)-Z_i^{j+1} \tag{22}$$

Substituting (8) in (22) gives in view of (21) and (22) we obtain after collecting like term in V_{i-1} ,

$$\frac{\sigma^2S^2}{2}\left[\frac{V_{i+1}^j-2V_i^j+V_{i-1}^j}{(\Delta S)^2}\right]+rS_i\left[\frac{V_{i+1}^j-V_{i-1}^j}{2\Delta S}\right]-rV_i^j=\frac{2}{\Delta t}\left(V_i^{j+1}-V_i^j\right)-Z_i^{j+1}$$

That is

$$\frac{\sigma^2S_i^2}{2(\Delta S)^2}\left[V_{i+1}^j-2V_i^j+V_{i-1}^j\right]+\frac{rS_i}{2\Delta S}\left[V_{i+1}^j-V_{i-1}^j\right]-rV_i^j=\frac{2}{\Delta t}\left(V_i^{j+1}-V_i^j\right)+Z_i^{j+1}$$

Collecting like terms in of V_{i-1},V_i *and* V_{i+1} and simplifying gives

$$Z_i^j=V^j{}_{i-1}\left[\frac{\Delta t\sigma^2S_i^2}{2(\Delta S)^2}-\frac{r\Delta tS_i}{2\Delta S}\right]+V_i^j\left[\frac{2\Delta t}{\Delta t}-\frac{\Delta t\sigma^2S_i^2}{(\Delta S)^2}-r\Delta t\right]+V^j{}_{i+1}\left[\frac{\Delta t\sigma^2S_i^2}{2(\Delta S)^2}+\frac{r\Delta tS_i}{2\Delta S}\right]$$

and

$$Z_i^{j+1} = V_{i-1}^{\ j+1}\left[\frac{\Delta t\sigma^2 S_i^{\ 2}}{2(\Delta S)^2} - \frac{r\Delta t S_i}{2\Delta S}\right] + V_i^{j+1}\left[\frac{2\Delta t}{\Delta t} - \frac{\Delta t\sigma^2 S_i^{\ 2}}{(\Delta S)^2} - r\Delta t\right] + V_{i+1}^{\ j+1}\left[\frac{\Delta t\sigma^2 S_i^{\ 2}}{2(\Delta S)^2} + \frac{r\Delta t S_i}{2\Delta S}\right]$$

Using (8) in (20) solving simultaneously and taking the average of these two equations we obtain

$$\left.\begin{aligned} &V_{i-1}^{\ j}\left[\frac{\Delta t\sigma^2 S_i^{\ 2}}{4(\Delta S)^2} - \frac{rS_i\Delta t S_i}{4\Delta S}\right] + V_i^{\ j}\left[1 - \left(\frac{\Delta t\sigma^2 S_i^{\ 2}}{2(\Delta S)^2} - \frac{r\Delta t}{2}\right)\right] + V_{i+1}^{\ j}\left[\frac{\Delta t\sigma^2 S_i^{\ 2}}{4(\Delta S)^2} + \frac{rS_i\Delta t S_i}{4\Delta S}\right] \\ &= V_{i-1}^{\ j+1}\left[\frac{\Delta t\sigma^2 S_i^{\ 2}}{4(\Delta S)^2} - \frac{rS_i\Delta t S_i}{4\Delta S}\right] + V_i^{\ j+1}\left[1 - \frac{\Delta t\sigma^2 S_i^{\ 2}}{2(\Delta S)^2} + \frac{r\Delta t}{2}\right] + V_{i+1}^{\ j+1}\left[\frac{\Delta t\sigma^2 S_i^{\ 2}}{4(\Delta S)^2} + \frac{rS_i\Delta t S_i}{4\Delta S}\right] \end{aligned}\right\} \quad (23)$$

The expressions inside the square brackets will be replaced with the coefficients a, b, c. The following equations obtained.

$$aV_{i-1}^{\ j} + bV_i^{\ j} + cV_{i+1}^{\ j} = aV_{i-1}^{\ j+1} + bV_i^{\ j+1} + cV_{i+1}^{\ j+1} \quad (24)$$

Where

$a_i = \frac{\Delta t}{4}\left((\sigma^2 S_i^2) - rS_i\right)$, $b_i = -\frac{\Delta t}{2}\left(\sigma^2 S_i^2 - r\right)$ and $c_i = -\frac{\Delta t}{4}\left((\sigma^2 S_i^2) - rS_i\right)$

a_i, b_i, c_i are random variables; $i = 0,1,...,M$.

Equation (23) can now be represented in matrix form as follows

$XV^j = YV^{j+1}, j = 0,1,2,...$

$\Rightarrow u^j = X^{-1}YV^{j+1}$

where $V^j = (V_{1,i}, V_{2,i}, V_{3,i}, ..., V_{m,i})^T$, ($V^i$ *is an* $m \times n$)

$$\begin{bmatrix} b & c & 0 & \cdots & 0 \\ -a & b & c & \cdots & 0 \\ 0 & -a & b & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \cdots & & b_{M-1} \end{bmatrix} \begin{bmatrix} V_1^j \\ V_2^j \\ V_3^j \\ \vdots \\ V_{M-1}^j \end{bmatrix} = \begin{bmatrix} X_1^{\ j} \\ X_2^{\ j} \\ X_3^j \\ \vdots \\ X_{M-1}^j \end{bmatrix}$$

$$\begin{bmatrix} b & c & 0 & \cdots & 0 \\ a & b & c & \cdots & 0 \\ 0 & a & b & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \cdots & & b_{M-1} \end{bmatrix} \begin{bmatrix} V_1^{j+1} \\ V_2^{j+1} \\ V_3^{j+1} \\ \vdots \\ V_{M-1}^{j+1} \end{bmatrix} = \begin{bmatrix} Y_1^{j+1} \\ Y_2^{j+1} \\ Y_3^{j+1} \\ \vdots \\ Y_{M-1}^{j+1} \end{bmatrix}$$

3.1 RESULTS AND DISCUSSION

In this Chapter we present computational results obtained by applying Section 2.1, for European Call Option with $S_0 = 40, K = 25, r = 0.2, t = 1$

Table 3.1 : BS versus CN comparison of Call option with Error differences

Volatility	Black-Scholes close form prices	Crank-Nicolson approximate prices	Absolute Error	%Error
0.25	19.54	19.54	0.00	0.00%
0.3	19.57	19.56	0.01	0.05%
0.35	19.64	19.59	0.05	0.25%
0.4	19.75	19.65	0.10	0.51%
0.45	19.91	19.72	0.19	0.95%
0.5	20.12	19.81	0.31	1.54%
0.55	20.36	19.91	0.45	2.21%
0.6	20.63	20.03	0.60	2.91%
0.65	20.94	20.17	0.77	3.68%

Table 3.1 describes the call prices of Black-Scholes model and Crank-Nicolson numerical solutions. Black-Scholes is exact for vanillas, so Crank-Nicolson's job is to match it. The table shows CN is fine for low volatility but degrades as volatility rises. In real markets where volatility clusters and jumps, especially in Nigerian equities, you either refine the grid or you accept mispricing that will bite during stress. Error grows with volatility : CN is underpricing relative to BS, and the underpricing gets worse as volatility rises.

Table 3.2 : BS versus CN comparison of Put option with Error differences: $S_0 = 40, K = 100, r = 0.2, t = 1$

Volatility (σ)	Black-Scholes close form prices	Crank-Nicolson approximate prices	Absolute Error	%Error
0.25	41.88	41.88	0.00	0.00%
0.3	41.92	41.90	0.02	0.05%
0.35	42.02	41.95	0.07	0.17%
0.4	42.20	42.03	0.17	0.40%
0.45	42.47	42.14	0.33	0.78%
0.5	42.83	42.29	0.54	1.26%
0.55	43.26	42.46	0.80	2.85%
0.6	43.77	42.64	0.13	2.58%
0.65	44.33	42.84	0.49	3.36%

In Table 3.2 , CN is stable and second order accurate, but "2nd-order" only helps if ΔS and Δt are small enough. High volatility needs a much finer grid. You are seeing the cost of using a normal market grid during volatility clustering.

However, Call versus Put differences; puts here are deeper value , so absolute error is bigger. But puts are also more sensitive to left-tail risk. Underpricing puts represents underestimating crash risk. In Nigeria where policy shocks and FX moves create left-tail events, that's dangerous.

Table 3.3 : Head-to- head comparison of Call and Put Options

Volatility (σ)	**Absolute Error**	**Put Absolute Error**	**Call %Error**	**Put %Error**	Which is worse
0.25	0.00	0.00	0.00%	0.00%	Same
0.3	0.01	0.02	0.05%	0.05%	Put slightly
0.35	0.05	0.07	0.25%	0.17%	Call% worse
0.4	0.10	0.17	0.51%	0.40%	Call% worse
0.45	0.19	0.33	0.95%	0.78%	Call% worse
0.5	0.31	0.54	1.54%	1.26%	Call% worse
0.55	0.45	0.80	2.21%	2.85%	Call% worse
0.6	0.60	0.13	2.91%	2.58%	Call% worse
0.65	0.77	0.49	3.68%	3.36%	Call% worse

As it can be seen in Table 3.3, both CN schemes break down as volatility rises. Error grows roughly quadratically with volatility. That's expected : Crank-Nicolson's truncation error for the BS PDE is $0\left(\Delta t^2, \Delta S^2, \sigma^2\right)$. When volatility jumps from 25% to 65%, the σ^2 term goes up 6.76 times. However, when absolute error of Puts is greater than Calls, at V=0.65, Put error=0.77. Almost double, look at BS levels: put are approximately 44, calls are approximately 21. The puts are more expensive, likely deeper ITM or longer maturity. CN's absolute error scales with the option level , Relative error when Calls is greater than puts.Call% error hits 3.68% versus put% error 3.36% at V=0.65. Calls are more sensitive to discretization error in% terms here. Intuition: ATM/OTM calls have higher gamma than deep ITM puts. CN smooths out gamma , so the cheaper ATM call gets hit harder proportionally, see columns 4 and 5 respectively. In all, CN works great in calm markets. But because error ,volatility clustering in Nigerian equities exposes the weakness. Calls get hit worse in % terms, puts get hit worse in naira terms. Either way, you were mispriced when it matters.

Table 3.4: Quantile Function for Call and Put Options

Percentile	Call BS	Call CN	Put BS	Put CN
50th	19.91	19.72	42.47	42.14
95th	20.78	20.10	44.05	42.74
100th	20.94	20.17	44.33	42.84

Table 3.4 matters because risk teams don't care about averaging error. They care about the 95th, 99th percentile errors because that's what breaches VaR. The table shows CN's error is highest exactly where it is most sensitive.

3.1.1 Model Analysis and Physical Characteristics

Let $V_{BS}^{C}\left(S,K,T,\sigma,\mathrm{r}\right)$ be the BS call price, V_{CN}^{C} be CN price on grid with steps $\Delta S,\Delta t$.

Assume $r=0$ for simplicity , results extend, hence we state and prove the following theorems.

Theorem 3.1 (Non-negativity and Monotonicity of Call Error). For a uniform CN grid with sufficiently smooth terminal condition, the discretization error $E^{C}=V_{CN}^{C}$ satisfies

1. $E^{C}\geq 0$ for all $\sigma>0$,i.e, CN underprices calls

2. E^{C} is increasing in σ^{2} .

Proof: The BS PDE is $\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^{2}S^{2}\frac{\partial^{2}V}{\partial S^{2}}=0$,CN discretization gives truncation error $\tau=\frac{1}{12}\sigma^{2}S^{2}\Delta S^{2}\frac{\partial^{2}V_{BS}^{C}}{\partial S^{4}}+0\left(\Delta t^{2}\right)$. For calls, $\frac{\partial^{2}V_{BS}^{C}}{\partial S4}>0$ in the gamma-convex region, so $\tau\geq 0$ .The error equation for $e=V_{BS}^{C}-V_{CN}^{C}$ is CN with source $\tau\geq 0$ and zero terminal data. By the discrete maximum principle, $e\geq 0.$ Since $\tau\alpha\sigma^{2},e$ increases in σ^{2} .

Theorem 3.2 (Call Error Lower Bound). There exists constant $C_1 > 0$ depending on K, T such that for small $\Delta S, E^C \geq C_1 \sigma^2 \Delta S^2 \Gamma_{ATM}$ where Γ_{ATM} is Black-Scholes gamma at-the-money.

Proof: Using the Green's function representation, $e_i^n \geq T \min \tau$. Since $\tau \geq \frac{e}{12} \sigma^2 \Delta S^2 \Gamma / S^2$ and Γ_{ATM} peaks ATM, setting $C_1 = eT/12$ gives the bound. This matches data at $\sigma = 0.65, E^C = 0.77$ is consistent with $\sigma^2 \Delta S^2 \Gamma$ scaling.

Theorem 3.3 (Put Error Dominates Call Error in Absolute Terms). For In-The-Money (ITM) puts with same K, T, σ as calls , if $V_{BS}^P > V_{BS}^C$, then $E^P > E^C$.

Proof: CN preserves put-call parity exactly because $S - Ke^{-rT}$ is linear. Thus $E^C - E^P = 0$ from parity alone. The extra error for puts comes from boundary behavior at $S = 0$. For puts, $\frac{\partial^2 V_{BS}^P}{\partial S^4}$ [illegible] as $S \to \to 0$, so $\tau^P > \tau^C$ .Hence $E^P > E^C$. Data confirms: $E^P = 1.49 > 0.77 = E^C at\ \sigma = 0.65$.

Theorem 3.4 (Put Error Bound and Tail Sensitivity). For puts, $E^P \geq C_2 \sigma^2 \Delta S^2 / K$ where $C_2 > 0$. Moreover, $\frac{\partial E^P}{\partial \sigma} > \frac{\partial E^C}{\partial \sigma}$ when $S < K$.

Proof: As $S \to 0, \frac{\partial^2 V_{BS}^P}{\partial S^4}$ [illegible] $\frac{2}{K\sigma^2 \sqrt{2\pi (T-t)^3}}$. Substituting into τ gives $\tau^P \geq \frac{\Delta S^2 S^2}{6K\sigma\sqrt{2\pi (T-t)^3}}$ .Propagation yields $E^P \geq C_2 \sigma^2 \Delta S^2 / K$. Differentiating gives $\partial E^P / \partial \sigma > \partial E^C / \partial \sigma$ since V_{SSSS}^{put} decays slower in σ. This explains faster error growth for puts in the tables.

The four theorems show CN bias is structural, quadratic in σ clusters at 0.5-0.7, grid spacing must reduce by approximately 2.4 times relative to 25% volatility calibrations to maintain 50bps accuracy.

4.1 CONCLUSION

This study demonstrates that Crank-Nicolson discretization error is economically significant under high-volatility conditions prevalent in Nigerian cement stocks. For European calls, CN underpricing reaches 3.36%. The error distribution is right-skewed and increases approximately with σ^2 , consistent with truncation error in the diffusion term. Quantile analysis confirms that tail-risk scenarios are most affected , with 95th- percentile pricing error of 0.68 and 1.31 for calls and puts respectively. These findings have three implications : (i) model validation frameworks must stress-test grids at $\sigma > 50\%$, not just ATM

25% volatility (ii) market makers using CN for NGX options face adverse selection when volatility spikes, as they quote too cheap (iii) VaR models repricing books with CN will breach back-tests during volatility clusters. The theorems proved establish that the bias is structural, not random, and provide lower bounds for error given grid size. We recommend adaptive mesh refinement where $\Delta S \alpha 1/\sigma$ or switching to high-order compact schemes for $\sigma > 40\%$. Future work should extend this analysis to American options and stochastic volatility models calibrated to NGX data.

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