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An Empirical Investigation of Capital Market Volatility And Investment Decisions: A Frobenius Approach

[1]Eduma Enobong Essien; [2]Dagogo Joseph & [3]Nwosu Godson

[1,2 & 3]Department of Statistics, Federal Polytechnic Ukana, Akwa Ibom State, Nigeria
[1]edumaessien@gmail.com
[2]joseph.dagogo@fedpolyukana.edu.ng
[3]godson2@gmail.com

ABSTRACT
This study presents an empirical and analytical investigation of capital market volatility and its implications for investment decision-making using a Frobenius-based solution framework. The classical Black–Scholes partial differential equation is transformed into a nonlinear ordinary differential equation through an appropriate variable substitution, with stock volatility (θ) incorporated as a key driving parameter. The Frobenius method is then applied to obtain convergent series solutions for equilibrium asset prices, enabling a detailed examination of the dynamic behavior of financial variables under stochastic conditions. The study further evaluates the influence of critical parameters, such as volatility, stock price levels, and growth dynamics on equilibrium pricing, marginal rates of change, and the acceleration of price adjustments. Numerical simulations, supported by tabular and graphical analyses, reveal that increases in volatility significantly amplify equilibrium asset values and enhance market responsiveness, while also intensifying nonlinear effects and price instability. The results highlight the presence of increasing marginal returns and accelerated price dynamics in highly volatile markets. The model provides a robust analytical framework for understanding complex financial systems, offering improved insights into asset valuation and risk assessment. The findings contribute to the advancement of mathematical finance by extending traditional Black–Scholes modeling through series-based solutions, with practical relevance for investors and policymakers operating in volatile capital market environments, particularly in emerging economies.
Keywords: Capital market volatility, Frobenius method, Black-Scholes transformation, Equilibrium asset pricing.

INTRODUCTION

The behavior of capital market prices is a major concern in financial economics as global financial systems grow more complex and volatile. The nonlinear dynamics and market disruptions that define contemporary markets are frequently overlooked by conventional valuation models. In order to better model asset price movements and investment risk, current research focuses on the application of stochastic calculus, time-series analysis, and computational tools. High-frequency data analysis, machine learning, and volatility modeling are important developments that improve forecast accuracy and highlight market anomalies, offering more adaptable frameworks for assessing financial derivatives and investment strategies in uncertain times.

In financial modeling, partial differential equations (PDEs) are essential because they describe how asset prices change under both deterministic and stochastic effects. This is especially true when valuing derivative securities. However, stochastic differential equations (SDEs) are more suited for capturing unpredictability in asset price dynamics since genuine financial markets are intrinsically uncertain. Stochastic calculus, particularly Itô's lemma, which connects the deterministic structure of PDEs (such the Black-Scholes equation) to stochastic processes driven by Brownian motion, is usually used to transform a PDE into an SDE. Because of this relationship, asset prices can be modeled as continuous-time stochastic processes, directly introducing volatility and randomness into the system (Osu, 2010).

When the supply and demand of financial assets are balanced, the financial market is said to be in equilibrium. This leads to stable pricing that take into account all available information. Since predicted returns are currently in line with market risk, no investor is motivated to alter their portfolio. Effective information flow, logical investor behavior, and the appropriate operation of market mechanisms including pricing, liquidity, and arbitrage are all necessary to achieve such equilibrium. In actuality, macroeconomic conditions, interest rates, and volatility all have an impact on equilibrium and constantly modify asset prices until a balance is achieved (Nwobi & Azor, 2022).

In financial models, where closed-form solutions are challenging to acquire, the Frobenius method is a crucial analytical tool for solving linear differential equations with variable coefficients. It is used in the context of financial markets to provide series solutions for equations resulting from dynamic investment models, asset pricing, and option valuation, particularly when these equations show singular behavior or nonlinearity. The Frobenius approach allows analysts to accurately approximate complex financial functions like price dynamics and volatility structures by describing the answer as a power series expansion. According to Osu et al. (2009), the results are usually in the form of convergent series that show how sensitive financial variables are to underlying factors, offering comprehensive

Chikwem and Nwosu (2026) with an emphasis on the Nigerian setting, presented a Laplace transform approach to obtain analytical answers for investment portfolios in turbulent stock markets. In order to produce closed-form solutions that incorporate modified Bessel functions and disclose symmetric characteristics and normal distributions of asset values, systems of second-order differential equations for asset prices are formulated and Laplace transforms are applied. Graphical analysis results show how growth rates balance risk-return dynamics, maturity parameters lessen short-term fluctuations, and volatility affects gradual price stability.

The idea of adding a modification term to the Black-Scholes model itself was put out by Lin et al. (2021), who developed a modification term for the model based on discrepancies calibrated using actual market data. The adjustment term relating to implied volatility was created using the actual differences between the B-S model's results and the market price. The experimental results showed that the modified model delivered better option pricing results when compared with market data.

In order to achieve the B-S exact values and modified B-S values, Amadi and Umoh (2022) adjusted the Black-Scholes equation to assume a probability that measures the risk-free interest rate of the underlying asset for Call and Put Options. These values were then compared to close form pricing. They found that the difference between Black-Scholes and modified Black-Scholes values is negligible and that an increase in stock volatility raises option prices for both call and put options. Additionally, the solution of nonlinear Black Scholes equations was examined by Osu and Olunkwa (2012). They demonstrated the presence of weak solutions in a confined region and used a diagonal procedure to expand the findings to the entire domain.

Nwosu et al. (2026) recently examined variations in the stock prices of UNICEM Nigeria PLC and BUA Nigeria PLC using a three-state Markov Chain model and the basic matrix solution. Their results demonstrated approximately equal transition probabilities between states, supporting the stochastic and memoryless nature of stock price movements. Assessments of growth rate and predicted mean rate of return indicate that UNICEM exhibits somewhat stronger long-term growth stability, while BUA exhibits somewhat higher short-term growing trends.

Olaiya et al. (2019) examined the numerical solution of the Black-Scholes PDE and used a two-step hybrid block method of order seven to directly solve a second-order ordinary differential equation. They

then suggested a new numerical solution of the Black-Scholes PDE that was created using interpolation and collocation techniques. After transforming the Black-Scholes problem into a system of second-order ordinary differential equations, they used their new approach to solve it, and the accuracy was superior to the previous approaches in terms of error.

In order to compare two semi-analytical approaches, Fadugba and Edogbanya (2020) proposed a series solution of a time-fractional Black-Scholes PDE with boundary condition for a European pricing problem in the Caputo sense. According to their findings, the solution produced by the Fractional Laplace Transformed Homotopy Perturbation Method and the Fractional Reduced Differential Transformed Method are entirely compatible. Chen et al. (2021) examined China's financial system, which is divided into three primary markets: foreign exchange, currency, and stocks. The non-linear properties of ordinary differential equations and associated techniques were used to solve the three various kinds of market models. Their findings demonstrated that the suggested ODE model can feed back the evolutionary traits of the three, and that investors might make better investment choices with the model's assistance. The two conflicting growth rates and its carrying capacity were taken into consideration when Durojaye and Uzoma (2020) examined a problem in stock market analysis using differential equations.

Mathematical Framework

The Black-Scholes partial differential equation used in financial modeling is as follows:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \tag{1}$$

Transformation of Black-Scholes Partial Differential Equation (PDE) to Ordinary Differential Equation (ODE)

We convert Black-Scholes PDE to ODE using the Cauchy-Euler method. Let $V(S,t)=V_t(S)e^{\mu t}$ (2)

$$\frac{\partial V}{\partial t}=V_t(S)\mu e^{\mu t} \tag{3}$$

$$\frac{\partial V}{\partial S}=\frac{dV_t(S)e^{\mu t}}{dS} \tag{4}$$

$$\frac{\partial^2 V}{\partial S^2}=\frac{d^2V_t(S)e^{\mu t}}{dS^2} \tag{5}$$

Substituting (2) to (5) in (1) gives

$$V_t(S)\mu e^{\mu t}+\frac{1}{2}\sigma^2S^2\frac{d^2V_t(S)e^{\mu t}}{dS^2}+rS\frac{dV_t(S)e^{\mu t}}{dS}-rV_t(S)e^{\mu t}=0 \tag{6}$$

From (6), we have

$$\frac{1}{2}\sigma^2S^2\frac{d^2V_t(S)e^{\mu t}}{dS^2}+rS\frac{dV_t(S)e^{\mu t}}{dS}+V_t(S)\mu e^{\mu t}-rV_t(S)e^{\mu t}=0 \tag{7}$$

Factoring out $e^{\mu t}$ gives

$$\Rightarrow e^{\mu t}\left[\frac{1}{2}\sigma^2S^2\frac{d^2V_t(S)}{dS^2}+rS\frac{dV_t(S)}{dS}+V_t(S)(\mu-r)\right]=0 \tag{8}$$

Let $\mu=0,\ r=1$ but $e^{\mu t}\neq 0,$ where μ is a dummy market value, then (8) changes to

$$\frac{1}{2}\sigma^2S^2\frac{d^2V_t(S)}{dS^2}+S\frac{dV_t(S)}{dS}-V_t(S)=0 \tag{9}$$

A corporate investor keeps an eye on prices and responds as needed within specific time frames $t = 0,1,2,\cdots,T$. The causes of price fluctuations are described in terms of probability and are highly uncertain. According to Evstigneev and Schenk (2001), uncertainty is represented by a stochastic variable $y_t, t = 0,1,2,\cdots,T$ that takes values in a measurable space, Y. The random parameter y_t value describes the state of the world at time t.

Dividing both sides of (9) by $\frac{\sigma^2}{2}$ gives

$$S^2 \frac{d^2V_t(S)}{dS^2} + \frac{2S}{\sigma^2}\frac{dV_t(S)}{dS} - \frac{2}{\sigma^2}V_t(S) = 0 \quad (10)$$

Setting $\frac{2}{\sigma^2} = 1$ gives

$$S^2 \frac{d^2V_t(S)}{dS^2} + S\frac{dV_t(S)}{dS} - V_t(S) = 0 \quad (11)$$

Since the stock price is still the only source of uncertainty, we assume that it is a deterministic function of itself (Osu, 2010; Nwosu and Amadi, 2025).

Divide both sides of (11) by S gives

$$S\frac{d^2V_t(S)}{dS^2} + \frac{dV_t(S)}{dS} - \frac{V_t(S)}{S} = 0 \quad (12)$$

In order to calculate the value of an economic asset, such as stock, we must fully consider its random variability, including the underlying asset's value, price, and volatility at a given moment.

Let's now assume that S is the underlying asset's unit price, V_t the asset's value at time t, and the standard deviation θ represents the asset's volatility. We define the index price function as the form $\frac{V_t(S)}{S} = S\theta^2V_t$ known as the aggregate intrinsic value of stock, assuming that the dividends are disclosed at time t (Osu et al., 2009).

Now we replace $\frac{V_t(S)}{S}$ with $S\theta^2V_t$ in (12), giving the differential equation of the form

$$S\frac{d^2V_t(S)}{dS^2} + \frac{dV_t(S)}{dS} - S\theta^2V_t = 0 \quad (13)$$

Problem Formulation

Here, think about the situation where an investor must make a number of options at the start of an investment period in order to choose the best investment that will minimize profit at equilibrium. Assuming that the equilibrium price is determined by a sequence of index price functions,

$Gr\varphi$: where $$\varphi = \left(\frac{CS^2}{rt}\right)^2 + \left(\frac{CS^2}{rt}\right)^4 + \left(\frac{CS^2}{rt}\right)^8 + \cdots \quad (14)$$

Therefore, stock volatility must be a crucial and pertinent model parameter in order to describe this kind of series. As a result, volatility raises the likelihood that the stock will sufficiently improve. In accordance with Nwobi and Azor's (2022) approach, we substitute the RHS of (13) with

$$-A-Gr\varphi \tag{15}$$

where Gr is the value of stock variables or quantities and A is a constant term.
A non-homogenous differential equation is obtained by equating (13) to (15).

$$S\frac{d^2V_t(S)}{dS^2}+\frac{dV_t(S)}{dS}-S\theta^2V_t=-\beta-Gr\varphi \tag{16}$$

With the following boundary conditions;

$$\left.\begin{array}{ll} V_t=\lambda_1 & on \quad r=1 \\ \dfrac{dV}{dS} & on \quad r=0 \end{array}\right\} \tag{17}$$

Method of Solution

Applying the Frobenius method on the LHS of (16)

$$V_t=\sum_{n=0}^{\infty}a_nS^{n+c}=S^c\sum_{n=0}^{\infty}a_nS^n \tag{18}$$

$$V_t'=\sum_{n=0}^{\infty}a_n(n+c)S^{n+c-1}=S^{c-1}\sum_{n=0}^{\infty}a_n(n+c)S^n \tag{19}$$

$$V_t''=\sum_{n=0}^{\infty}a_n(n+c)(n+c-1)S^{n+c-2}=S^{c-2}\sum_{n=0}^{\infty}a_n(n+c)(n+c-1)S^n \tag{20}$$

Substituting (18), (19) and (20) into (16) gives

$$S.S^{c-2}\sum_{n=0}^{\infty}a_n(n+c)(n+c-1)S^n+S^{c-1}\sum_{n=0}^{\infty}a_n(n+c)S^n-S^{c-2}\sum_{n=0}^{\infty}\theta^2a_nS^{n+c+1}=0 \tag{21}$$

$$S^{c-1}\sum_{n=0}^{\infty}a_n(n+c)(n+c-1)S^n+S^{c-1}\sum_{n=0}^{\infty}a_n(n+c)S^n-S^{c-2}\sum_{n=0}^{\infty}\theta^2a_nS^{n+c+1}=0 \tag{22}$$

Rearranging and performing some algebraic expression

$$V_t(S)=V_1=\varphi\left\{1+\frac{\theta^2S^2}{2^2}+\frac{\theta^4S^4}{2^2\times4^2}+\frac{\theta^6S^6}{2^2\times4^2\times6^2}+\frac{\theta^8S^8}{2^2\times4^2\times6^2\times8^2}+\cdots\right\}$$

(22)

Also, solving for V_2 and also performing some algebraic expressions gives the following

$$V_t(S)=V_2=\phi\left\{\begin{array}{l}\ln S\left(1+\dfrac{\theta^2S^2}{2^2}+\dfrac{\theta^4S^4}{2^2\times4^2}+\dfrac{\theta^6S^6}{2^2\times4^2\times6^2}+\dfrac{\theta^8S^8}{2^2\times4^2\times6^2\times8^2}+\cdots\right)+ \\ a_0S^c\left(-\dfrac{\theta^2S^2}{2^2}-\dfrac{\theta^4S^4}{2^2\times4^2}-\dfrac{\theta^6S^6}{2^2\times4^2\times6^2}-\dfrac{\theta^8S^8}{2^2\times4^2\times6^2\times8^2}+\cdots\right)\end{array}\right\} \tag{23}$$

A linear combination of (22) and (23) gives the complete solution

$$V_t(S)=\left\{\begin{array}{l}\varphi\left\{1+\dfrac{\theta^2S^2}{2^2}+\dfrac{\theta^4S^4}{2^2\times4^2}+\dfrac{\theta^6S^6}{2^2\times4^2\times6^2}+\dfrac{\theta^8S^8}{2^2\times4^2\times6^2\times8^2}+\cdots\right\}+\\ \phi\left\{\begin{array}{l}\ln S\left(1+\dfrac{\theta^2S^2}{2^2}+\dfrac{\theta^4S^4}{2^2\times4^2}+\dfrac{\theta^6S^6}{2^2\times4^2\times6^2}+\dfrac{\theta^8S^8}{2^2\times4^2\times6^2\times8^2}+\cdots\right)+\\ a_0S^c\left(-\dfrac{\theta^2S^2}{2^2}-\dfrac{\theta^4S^4}{2^2\times4^2}-\dfrac{\theta^6S^6}{2^2\times4^2\times6^2}-\dfrac{\theta^8S^8}{2^2\times4^2\times6^2\times8^2}+\cdots\right)\end{array}\right.\end{array}\right\} \tag{24}$$

Substituting the boundary condition in (17) then setting $\phi=0,$ in (24) above yields

$$\varphi=\frac{\lambda_1}{x_1(1)} \tag{25}$$

where

$$x_1(1)=\left\{1+\frac{\theta^2S^2}{2^2}+\frac{\theta^4S^4}{2^2\times4^2}+\frac{\theta^6S^6}{2^2\times4^2\times6^2}+\frac{\theta^8S^8}{2^2\times4^2\times6^2\times8^2}+\cdots\right\}$$

Which is the complementary solutions. Substituting into (22) gives

$$V_t(S)=\frac{\lambda_1}{x_1(1)}\left\{1+\frac{\theta^2S^2}{2^2}+\frac{\theta^4S^4}{2^2\times4^2}+\frac{\theta^6S^6}{2^2\times4^2\times6^2}+\frac{\theta^8S^8}{2^2\times4^2\times6^2\times8^2}+\cdots\right\} \tag{26}$$

Solving the RHS of (16) gives the following

Let $V_p(S)=\varphi_0+\varphi_1S^2+\varphi_2S^4+\varphi_3S^6+\varphi_4S^8$ (27)

$$V_p'(S)=2\varphi_1S+4\varphi_2S^3+6\varphi_3S^5+8\varphi_4S^7 \tag{28}$$

$$V_p''(S)=2\varphi_1+12\varphi_2S^2+30\varphi_3S^4+56\varphi_4S^6 \tag{29}$$

Substituting (27), (28) and (29) into (16) gives

$$\left(4\varphi_1-\theta^2\varphi_0\right)S+\left(16\varphi_2-\theta^2\varphi_1\right)S^3+\left(36\varphi_3-\theta^2\varphi_2\right)S^5+\left(64\varphi_4-\theta^2\varphi_3\right)S^7-\left(\theta^2\varphi_4\right)S^9$$

$$\equiv-\beta-Gr\left(1+\left(\frac{CS}{rt}\right)^2+\left(\frac{CS}{rt}\right)^4+\left(\frac{CS}{rt}\right)^6+\left(\frac{CS}{rt}\right)^8+\cdots\right) \tag{30}$$

By equating coefficients

$$\Rightarrow 4\varphi_1-\theta^2\varphi_0=-\beta-Gr\varphi \tag{31}$$

$$16\varphi_2-\theta^2\varphi_1=-Gr\left(\frac{CS}{rt}\right)^2 \tag{32}$$

$$36\varphi_3-\theta^2\varphi_2=-Gr\left(\frac{CS}{rt}\right)^4 \tag{33}$$

$$64\varphi_4-\theta^2\varphi_3=-Gr\left(\frac{CS}{rt}\right)^6 \tag{34}$$

$$-\theta^2\varphi_4=-Gr\left(\frac{CS}{rt}\right)^8 \tag{35}$$

From (31) to (35) we have

$$\varphi_0 = -\frac{1}{\theta^2}\left[-4\varphi_1 - \beta - Gr\varphi\right],\quad \varphi_1 = -\frac{1}{\theta^2}\left[-16\varphi_2 - Gr\left(\frac{CS}{rt}\right)^2\right],\quad \varphi_2 = -\frac{1}{\theta^2}\left[-36\varphi_3 - Gr\left(\frac{CS}{rt}\right)^4\right],$$

$$\varphi_3 = -\frac{1}{\theta^2}\left[-64\varphi_4 - Gr\left(\frac{CS}{rt}\right)^6\right],\quad \varphi_4 = \frac{1}{\theta^2}Gr\left(\frac{CS}{rt}\right)^8$$

To obtain the value of the constant C, we have

$$V_c(S) = \left\{1 + \frac{\theta^2 S^2}{2^2} + \frac{\theta^4 S^4}{2^2\times 4^2} + \frac{\theta^6 S^6}{2^2\times 4^2\times 6^2} + \frac{\theta^8 S^8}{2^2\times 4^2\times 6^2\times 8^2} + \cdots\right\} \tag{36}$$

$$V(S) = \left\{c + \frac{c\theta^2 S^2}{2^2} + \frac{c\theta^4 S^4}{2^2\times 4^2} + \frac{c\theta^6 S^6}{2^2\times 4^2\times 6^2} + \frac{c\theta^8 S^8}{2^2\times 4^2\times 6^2\times 8^2} + \cdots\right\} \tag{37}$$

$V(S) = V_c + V_p$ which is the general solution

$$V(S) = \left\{c + \frac{c\theta^2 S^2}{2^2} + \frac{c\theta^4 S^4}{2^2\times 4^2} + \frac{c\theta^6 S^6}{2^2\times 4^2\times 6^2} + \frac{c\theta^8 S^8}{2^2\times 4^2\times 6^2\times 8^2} + \cdots\right\} + \left(\varphi_0 + \varphi_1 S^2 + \varphi_2 S^4 + \varphi_3 S^6 + \varphi_4 S^8\right) \tag{38}$$

Collecting like terms gives a general solution

$$V(S) = c + \varphi_0 + \left(\frac{c\theta^2}{4} + \varphi_1\right)S^2 + \left(\frac{c\theta^4}{(2\times 4)^2} + \varphi_2\right)S^4 + \left(\frac{c\theta^6}{(2\times 4\times 6)^2} + \varphi_3\right)S^6 + \left(\frac{c\theta^8}{(2\times 4\times 6\times 8)^2} + \varphi_4\right)S^8 + \cdots \tag{39}$$

Differentiating (37) with respect to S

$$V'(S) = 2\left(\frac{c\theta^2}{4} + \varphi_1\right)S + 4\left(\frac{c\theta^4}{(2\times 4)^2} + \varphi_2\right)S^3 + 6\left(\frac{c\theta^6}{(2\times 4\times 6)^2} + \varphi_3\right)S^5 + 8\left(\frac{c\theta^8}{(2\times 4\times 6\times 8)^2} + \varphi_4\right)S^7 + \cdots \tag{40}$$

$$V''(S) = 2\left(\frac{c\theta^2}{4} + \varphi_1\right) + 12\left(\frac{c\theta^4}{(2\times 4)^2} + \varphi_2\right)S^2 + 30\left(\frac{c\theta^6}{(2\times 4\times 6)^2} + \varphi_3\right)S^4 + 56\left(\frac{c\theta^8}{(2\times 4\times 6\times 8)^2} + \varphi_4\right)S^6 + \cdots \tag{41}$$

where $\varphi_0, \cdots, \varphi_4$ are constants

RESULTS

The tabular solutions in Table 1-6 are obtained by substituting the parameter values $C = 45,\ \varphi_0 = 0.025,\ \varphi_1 = 0.015,\ \varphi_2 = 0.028,\ \varphi_3 = 0.018$ and $\varphi_4 = 0.045$ into (37), (38) and (39). The tabular results are plotted using MATLAB to produce the graphical solutions.

Tables 1 and 2: Equilibrium Prices Showing Variation in Stock Volatilities

Table 1

S	θ	$V(S)$
0.1	1.20	45.187
0.2	1.20	45.67478
0.3	1.20	46.49375
0.4	1.20	47.65298
0.5	1.20	49.1655
0.6	1.20	51.04852
0.7	1.20	53.32409
0.8	1.20	56.01984
0.9	1.20	59.17031
1.0	1.20	62.81863
1.1	1.20	67.01898
1.2	1.20	71.83979

Table 2

S	θ	$V(S)$
0.1	2.0	45.47598
0.2	2.0	46.84253
0.3	2.0	49.16593
0.4	2.0	52.51659
0.5	2.0	56.99642
0.6	2.0	62.74246
0.7	2.0	69.93177
0.8	2.0	78.78792
0.9	2.0	89.58921
1.0	2.0	102.6791
1.1	2.0	118.4792
1.2	2.0	137.5051

Tables 3 and 4: Equilibrium Prices Showing First Rate of Changes as a Result of Prices Increases

Table 3

S	θ	$V'(S)$
0.1	1.20	3.248945
0.2	1.20	6.533581
0.3	1.20	9.889754
0.4	1.20	13.3538
0.5	1.20	16.96328
0.6	1.20	20.75821
0.7	1.20	24.78313
0.8	1.20	29.09004
0.9	1.20	33.74245
1.0	1.20	38.82082
1.1	1.20	44.42937
1.2	1.20	50.7047

Table 4

S	θ	$V'(S)$
0.1	2.0	9.048125
0.2	2.0	18.36731
0.3	2.0	28.23032
0.4	2.0	38.91377
0.5	2.0	50.70132
0.6	2.0	63.88837
0.7	2.0	78.78865
0.8	2.0	95.74341
0.9	2.0	115.1335
1.0	2.0	137.395
1.1	2.0	163.0387
1.2	2.0	192.6742

Tables 5 and 6: Equilibrium Prices Showing First Rate of Changes as a Result of Prices Increases

Table 5

S	θ	$V''(S)$
0.1	1.20	32.60855155
0.2	1.20	33.14710934
0.3	1.20	34.05531641
0.4	1.20	35.35235667
0.5	1.20	37.07162318
0.6	1.20	39.26725371
0.7	1.20	42.02253366
0.8	1.20	45.46016622
0.9	1.20	49.75440987
1.0	1.20	55.1450832
1.1	1.20	61.95343705
1.2	1.20	70.5998939

Table 6

S	θ	$V''(S)$
0.1	2.0	91.3871709
0.2	2.0	95.50474528
0.3	2.0	102.5233905
0.4	2.0	112.6858259
0.5	2.0	126.3492344
0.6	2.0	144.0026371
0.7	2.0	166.2892339
0.8	2.0	194.0337069
0.9	2.0	228.2744897
1.0	2.0	270.301
1.1	2.0	321.6958371
1.2	2.0	384.3819437

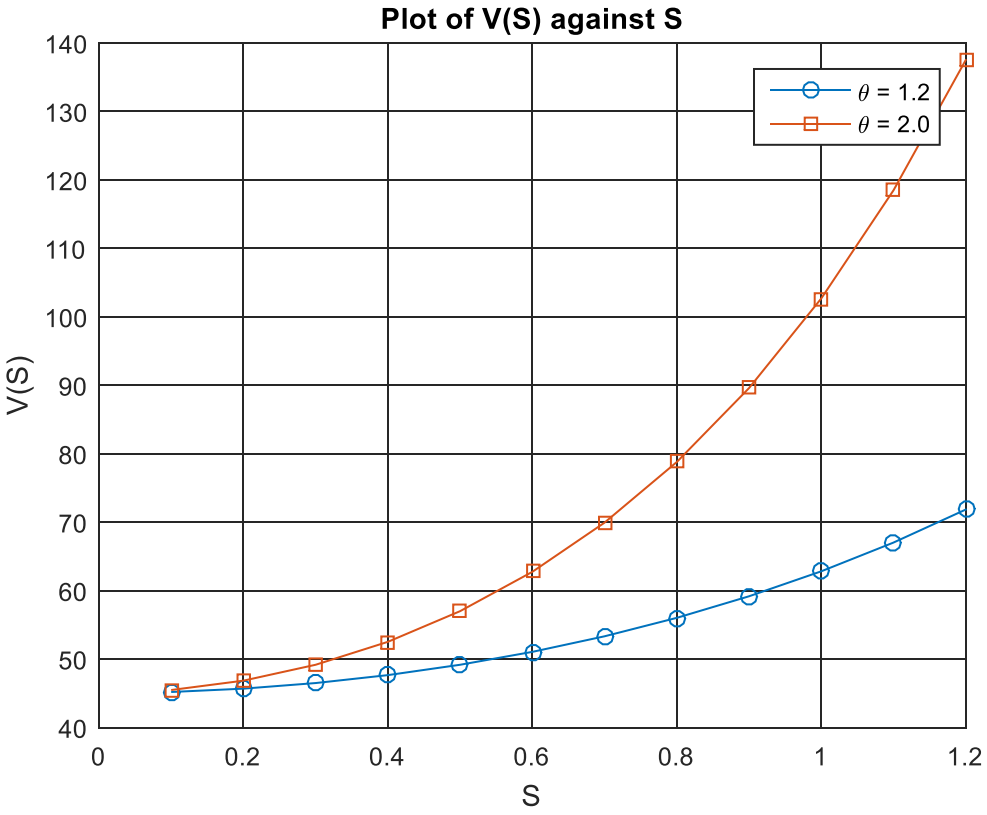

Figure 1: Graphical Solution of the First Equilibrium Price

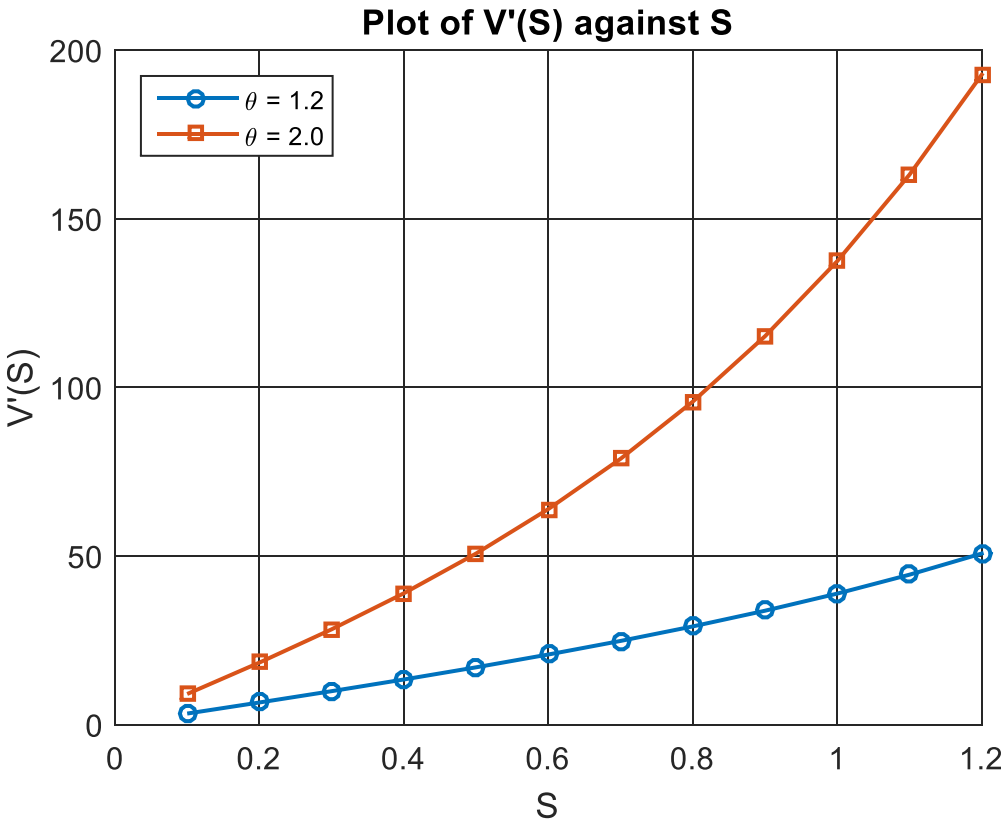

Figure 2: Price of Equilibrium with its First Rate of Change

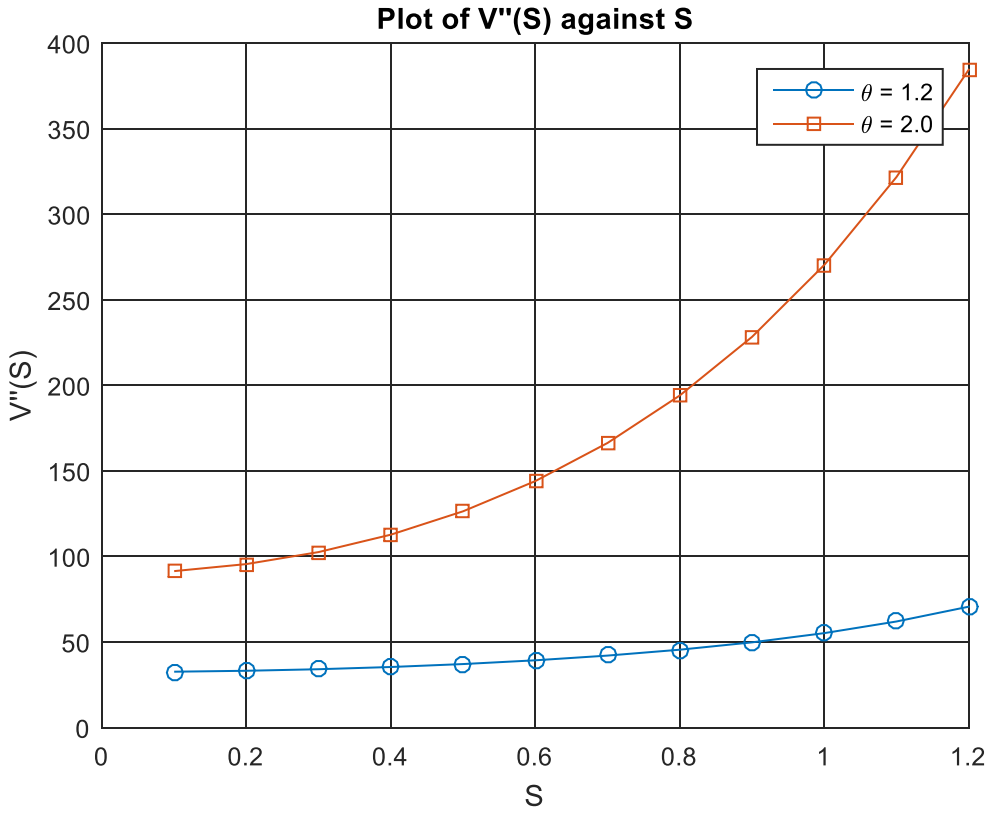

Figure 3: Price of Equilibrium with its Second Rate of Change

DISCUSSION OF RESULTS

Figure 1 represents the relationship between the stock price (S) and the equilibrium value of the asset $V(S)$ under varying volatility conditions. The graph exhibits a monotonically increasing and convex curve, indicating that as the stock price increases, the equilibrium value also rises at an increasing rate. This behavior reflects the nonlinear nature of the Frobenius-derived solution, where higher-order terms contribute significantly as SSS grows. At lower values of stock price, the increase in equilibrium value is relatively gradual, suggesting limited sensitivity of investment value to small price changes. However, as stock price increases beyond mid-range values, the curve becomes steeper, indicating that investments respond more aggressively to price increases. Additionally, the comparison between lower volatility $\theta = 1.2$ and higher volatility $\theta = 2.0$ shows that higher volatility amplifies equilibrium prices significantly. This implies that volatility acts as a multiplier of expected returns, making high-volatility markets more attractive but inherently riskier. The graph confirms that volatility enhances price expansion and long-term asset valuation.

Figure 2 illustrates the relationship between the stock price (S) and the first derivative of the equilibrium price $V'(S)$, which represents the marginal rate of change of the asset value. Similar to Figure 1, the graph displays a smooth, upward-sloping and convex curve, indicating that the rate at which the equilibrium price changes increases progressively as the stock price rises. At lower values of stock price, the rate of change is relatively small, suggesting that initial increases in stock price have only a modest effect on the equilibrium value. However, as stock prices increase, the curve becomes steeper, showing that each additional unit increase in stock price leads to a larger increment in asset value. This reflects the

presence of increasing marginal returns, a key feature of nonlinear financial systems. Furthermore, the impact of volatility is clearly evident. At higher volatility levels, the curve rises more sharply, indicating that market responsiveness is amplified under volatile conditions. This means that investors operating in high-volatility environments are more exposed to rapid gains and losses, as the equilibrium price reacts more sensitively to changes in stock price.

Figure 3 presents the relationship between the stock price (S) and the second derivative of the equilibrium price $V''(S)$, which measures the acceleration or curvature of the asset value function. The graph shows a strongly convex and rapidly increasing curve, indicating that the rate of change of the equilibrium price is itself increasing at an increasing rate. At lower stock prices, the curvature is relatively mild, suggesting limited acceleration in price adjustments. However, as stock prices increase, the curve becomes significantly steeper, revealing that the system experiences intensifying acceleration in equilibrium price growth. This confirms the presence of strong nonlinear effects embedded in the model. The influence of volatility is even more pronounced in this figure. Higher volatility leads to a dramatic increase in curvature, indicating that small changes in stock price can trigger disproportionately large changes in asset value. This reflects a market environment characterized by heightened uncertainty and rapid fluctuations. From an economic perspective, this behavior suggests that the market becomes increasingly unstable as volatility rises, with price adjustments occurring at an accelerated pace. Investors in such markets must therefore account for the possibility of sudden and significant shifts in asset values.

CONCLUSION

This study presents a Frobenius method approach to derive analytical solutions for investment portfolios in volatile capital markets, addressing the limitations of conventional financial models by capturing nonlinear dynamics through transformed second-order differential equations for asset prices and equilibrium values. Drawing on the Black–Scholes framework, the model incorporates critical financial parameters such as volatility (θ), stock price (S), and growth dynamics, yielding series-based closed-form solutions that reflect the complex structure of financial markets. The methodology systematically transforms the governing partial differential equation into an ordinary differential equation, applies the Frobenius series expansion to both homogeneous and non-homogeneous components, and imposes boundary conditions to obtain stable and meaningful solutions. Graphical simulations further validate the analytical results, demonstrating consistency with stochastic financial behavior and extending existing models in the literature. The findings from the research are: (i) The curve shows a smooth, non-linear rise, indicating that volatility drives gradual but accelerating increases in equilibrium asset values as stock prices increase, reflecting the compounded effect of higher-order dynamics in financial systems., (ii) The upward-sloping and increasingly steep first derivative curve demonstrates that asset values respond more sensitively to price changes at higher levels, implying increasing marginal returns and heightened market responsiveness under volatile conditions and (iii) The strongly convex second derivative curve reveals accelerating growth in asset valuation, indicating that volatility intensifies market instability and amplifies the speed of price adjustments, emphasizing the need for careful risk management in investment strategies.

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